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PROPERTIES OF SOME FAMILIES OF HYPERGEOMETRIC ORTHOGONAL POLYNOMIALS IN SEVERAL VARIABLES

J. F. VAN DIEJEN

Abstract. Limiting cases are studied of the Koornwinder-Macdonald multivariable generalization of the Askey-Wilson polynomials. We recover recently and not so recently introduced families of hypergeometric orthogonal polynomials in several variables consisting of multivariable Wilson, continuous Hahn and Jacobi type polynomials, respectively. For each class of polynomials we provide systems of difference (or differential) equations, recurrence relations, and expressions for the norms of the polynomials in terms of the norm of the constant polynomial.

1. Introduction

It is to date over a decade ago that Askey and Wilson released their famous memoir [AW], in which they introduced a four-parameter family of basic hypergeometric polynomials nowadays commonly referred to as the Askey-Wilson polynomials [GR]. These polynomials, which are defined explicitly in terms of a terminating ${}_4\phi_3$ series, have been shown to exhibit a number of interesting properties. Among other things, it was demonstrated that they satisfy a second order difference equation, a three-term recurrence relation, and that—in a suitable parameter regime—they constitute an orthogonal system with respect to an explicitly given positive weight function with support on a finite interval (or on the unit circle, depending on how the coordinates are chosen).

Many (basic) hypergeometric orthogonal polynomials studied in the literature arise as special (limiting) cases of the Askey-Wilson polynomials and have been collected in the so-called (q-)Askey scheme [AW, KS]. For instance, if the step size parameter of the difference equation is scaled to zero, then the Askey-Wilson polynomials go over in Jacobi polynomials: well-known classical hypergeometric orthogonal polynomials satisfying a second order differential equation instead of a difference equation. One may also consider the transition from orthogonal polynomials on a finite interval to

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orthogonal polynomials on a (semi-)infinite interval. This way one arrives at Wilson polynomials (semi-infinite interval) and at continuous Hahn polynomials (infinite interval).

The purpose of the present paper is to generalize this state of affairs from one to several variables. Starting point is a recently introduced multivariable generalization of the Askey-Wilson polynomials, found for special parameters by Macdonald [M2] and in full generality (involving five parameters) by Koornwinder [K]. By means of limiting transitions similar to those in the one-variable case, we arrive at multivariable Jacobi polynomials [V, De] (see also [BO] and reference therein) and at multivariable Wilson and continuous Hahn polynomials [D3].

The (q-)Askey scheme involves many more limits and special cases of the Askey-Wilson polynomials than those described above. For instance, one also considers transitions from certain polynomials in the scheme to similar polynomials with less parameters and transitions from polynomials with a continuous orthogonality measure to polynomials with a discrete orthogonality measure. Such transitions (or rather their multivariable analogues) will not be considered here. We refer instead to [D1, Sec. 5.2] for the transition from multivariable Askey-Wilson polynomials to Macdonald's q-Jack polynomials (i.e., multivariable q-ultraspherical polynomials) [M4] (as an example of a limit leading to similar polynomials but with less parameters), and to [SK] for the transition from multivariable Askey-Wilson polynomials to multivariable big and little q-Jacobi polynomials [S] (as an example of a limit leading to multivariable polynomials with a discrete orthogonality measure).

Whenever one is dealing with orthogonal polynomials an important question arises as to the explicit computation of the normalization constants converting the polynomials into an orthonormal system. For Jacobi polynomials calculating the orthonormalization constants boils down to the evaluation of (standard) beta integrals, whereas Askey-Wilson polynomials give rise to q-beta integrals. In the case of several variables one has to deal with Selberg type integrals (Jacobi case) and q-Selberg type integrals (Askey-Wilson case), respectively. For these multiple integrals explicit evaluations have been conjectured by Macdonald that were recently checked using techniques involving so-called shift operators [Op, HS, C1, M5]. (Roughly speaking these shift operators allow one to relate the values of the (q-)Selberg integral for different values of the parameters separated by unit shifts; the integral can then be solved, first for nonnegative integer-valued parameters by shifting the parameters to zero in which case the integrand becomes trivial, and then for arbitrary nonnegative parameters using an analyticity argument (viz. Carlson's theorem).)

Very recently, the author observed that Koornwinder's second order difference equation for the multivariable Askey-Wilson polynomials may be extended to a system of n (= number of variables) independent difference equations [D1] and that the polynomials also satisfy a system of n independent recurrence relations [D5]. (To date

a complete proof for these recurrence relations is only available for a self-dual four-parameter subfamily of the five-parameter multivariable Askey-Wilson polynomials.) It turns out that the recurrence relations, combined with the known evaluation for the norm of the unit polynomial (i.e., the constant term integral) [Gu, Ka], may also be used to verify Macdonald's formulas for the orthonormalization constants of the multivariable Askey-Wilson polynomials [D5]. Below, we will use these results to arrive at systems of difference (or differential) equations, recurrence relations and expressions for the orthonormalization constants, for all three limiting cases of the multivariable Askey-Wilson polynomials considered in this paper (Wilson, continuous Hahn and Jacobi type).

We would like to emphasize that much of the presented material admits a physical interpretation in terms of Calogero-Sutherland type exactly solvable quantum n-particle models related to classical root systems [OP] or their Ruijsenaars type difference versions [R1, R2, D2]. The point is that the second order differential equation for the multivariable Jacobi polynomials may be seen as the eigenvalue equation for a trigonometric quantum Calogero-Sutherland system related to the root system BC_n [OP]. From this viewpoint the second order difference equation for the multivariable Askey-Wilson polynomials corresponds to the eigenvalue equation for a Ruijsenaars type difference version of the BC_n-type quantum Calogero-Sutherland system [D4]. The transitions to the multivariable continuous Hahn and Wilson polynomials amount to rational limits leading to (the eigenfunctions of) similar difference versions of the A_{n-1}-type rational Calogero model with harmonic term (continuous Hahn case) and its $B(C)_n$-type counterpart (Wilson case) [D3]. For further details regarding these connections with the Calogero-Sutherland and Ruijsenaars type quantum integrable n-particle systems the reader is referred to [D2, D3, D4].

The material is organized as follows. First we define our families of multivariable (basic) hypergeometric polynomials in Section 2 and recall their second order difference equation (Askey-Wilson, Wilson, continuous Hahn type) or second order differential equation (Jacobi type) in Section 3. Next, in Section 4, limit transitions from the Askey-Wilson type family to the Wilson, continuous Hahn and Jacobi type families are discussed. We study the behavior of our recently introduced systems of difference equations and recurrence relations for the multivariable Askey-Wilson type polynomials with respect to these limits in Sections 5 and 6, respectively. The recurrence relations for the Wilson, continuous Hahn and Jacobi type polynomials thus obtained in Section 6 are then employed in Section 7 to derive explicit expressions for the (squared) norms of the corresponding polynomials in terms of the (squared) norm of the unit polynomial.

2. Multivariable (basic) hypergeometric polynomials

In this section multivariable versions of some orthogonal families of (basic) hypergeometric polynomials are characterized. The general idea of the construction (which is standard, see e.g. [V, M2, K, SK]) is to start with an algebra of (symmetric) polynomials $\mathcal{H}$ spanned by a basis of (symmetric) monomials $\{m_\lambda\}_{\lambda\in\Lambda}$, with the set Λ labeling the basis elements being partially ordered in such a way that for all $\lambda \in \Lambda$ the subspaces $\mathcal{H}_\lambda \equiv \mathrm{Span}\{m_\mu\}_{\mu\in\Lambda,\mu\leq\lambda}$ are finite-dimensional. It is furthermore assumed that the space $\mathcal{H}$ is endowed with an L^2 inner product $\langle\cdot,\cdot\rangle_\Delta$ characterized by a certain weight function Δ. To such a configuration we associate a basis $\{p_\lambda\}_{\lambda\in\Lambda}$ of $\mathcal{H}$ consisting of the polynomials p_λ, $\lambda \in \Lambda$, determined (uniquely) by the two conditions

i. $p_\lambda = m_\lambda + \sum_{\mu\in\Lambda,\mu<\lambda} c_{\lambda,\mu}\, m_\mu, \qquad c_{\lambda,\mu} \in \mathbb{C};$

ii. $\langle p_\lambda, m_\mu\rangle_\Delta = 0 \quad \text{if} \quad \mu < \lambda.$

In other words, the polynomial p_λ consists of the monomial m_λ minus its orthogonal projection with respect to the inner product $\langle\cdot,\cdot\rangle_\Delta$ onto the finite-dimensional subspace $\mathrm{Span}\{m_\mu\}_{\mu\in\Lambda,\mu<\lambda}$. By varying the concrete choices for the space $\mathcal{H}$, the basis $\{m_\lambda\}_{\lambda\in\Lambda}$ and the inner product $\langle\cdot,\cdot\rangle_\Delta$, we recover certain (previously introduced) multivariable generalizations of the Askey-Wilson, Wilson, continuous Hahn and Jacobi polynomials, respectively. Below we will specify the relevant data determining these families. The fact that in the case of one variable the corresponding polynomials p_λ indeed reduce to the well-known one-variable polynomials studied extensively in the literature is immediate from the weight function. The normalization for the polynomials is determined by the fact that (by definition) p_λ is monic in the sense that the coefficient of the leading monomial m_λ in p_λ is equal to one.

It turns out that in all of our cases the basis $\{m_\lambda\}_{\lambda\in\Lambda}$ can be conveniently expressed in terms of the monomial symmetric functions

$$m_{sym,\lambda}(z_1,\ldots,z_n) = \sum_{\mu\in S_n(\lambda)} z_1^{\mu_1}\cdots z_n^{\mu_n}, \qquad \lambda\in\Lambda, \tag{2.1}$$

where

$$\Lambda = \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}. \tag{2.2}$$

In (2.1) the summation is meant over the orbit of λ under the action of the permutation group S_n (acting on the vector components $\lambda_1,\ldots,\lambda_n$). As partial order of the integral cone Λ (2.2) we will always take the dominance order defined by

$$\mu \leq \lambda \quad \text{iff} \quad \sum_{j=1}^m \mu_j \leq \sum_{j=1}^m \lambda_j \ \text{ for } \ m = 1,\ldots,n \tag{2.3}$$

(and $\mu < \lambda$ iff $\mu \leq \lambda$ and $\mu \neq \lambda$).

Note: In order to avoid confusion between the various families we will often equip the polynomials and other objects of interest with the superscripts 'AW', 'W', 'cH' or 'J' to indicate Askey-Wilson, Wilson, continuous Hahn or Jacobi type polynomials, respectively. Sometimes, however, these superscripts will be suppressed when discussing more general properties of the polynomials that hold simultaneously for all families.

2.1. **Askey-Wilson type.** To arrive at multivariable Askey-Wilson type polynomials one considers a space $\mathcal{H}^{AW}$ consisting of even and permutation invariant trigonometric polynomials. Specifically, the space $\mathcal{H}^{AW}$ is spanned by the monomials

$$m_\lambda^{AW}(x) = m_{sym,\lambda}(e^{i\alpha x_1} + e^{-i\alpha x_1}, \cdots, e^{i\alpha x_n} + e^{-i\alpha x_n}), \qquad \lambda \in \Lambda \tag{2.4}$$

(with Λ given by (2.2)). The relevant inner product on $\mathcal{H}^{AW}$ is determined by

$$\langle m_\lambda^{AW}, m_\mu^{AW}\rangle_{\Delta^{AW}} = \int_{-\pi/\alpha}^{\pi/\alpha}\cdots\int_{-\pi/\alpha}^{\pi/\alpha} m_\lambda^{AW}(x)\, \overline{m_\mu^{AW}(x)}\, \Delta^{AW}(x)\, dx_1\cdots dx_n, \tag{2.5}$$

with the weight function reading

$$\begin{aligned} \Delta^{AW}(x) \quad = \quad & \prod_{\substack{1\le j<k\le n\\ \varepsilon_1,\varepsilon_2=\pm 1}} \frac{(e^{i\alpha(\varepsilon_1 x_j+\varepsilon_2 x_k)};q)_\infty}{(t\, e^{i\alpha(\varepsilon_1 x_j+\varepsilon_2 x_k)};q)_\infty} \\ & \times \prod_{\substack{1\le j\le n\\ \varepsilon=\pm 1}} \frac{(e^{2i\alpha\varepsilon x_j};q)_\infty}{(t_0\, e^{i\alpha\varepsilon x_j},\, t_1\, e^{i\alpha\varepsilon x_j},\, t_2\, e^{i\alpha\varepsilon x_j},\, t_3\, e^{i\alpha\varepsilon x_j};\, q)_\infty}. \end{aligned} \tag{2.6}$$

Here $(a;q)_\infty \equiv \prod_{m=0}^{\infty}(1-aq^m)$, $(a_1,\ldots,a_r;q)_\infty \equiv (a_1;q)_\infty\cdots(a_r;q)_\infty$ and the parameters are assumed to satisfy the constraints

$$\alpha > 0, \quad 0<q<1, \quad -1\le t\le 1, \quad |t_r|\le 1 \ \ (r=0,1,2,3), \tag{2.7}$$

with possible non-real parameters t_r occurring in complex conjugate pairs and pairwise products of the t_r being $\neq 1$. For the weight function in (2.6) the polynomials p_λ determined by the Conditions *i.* and *ii.* (above) were introduced by Macdonald [M2] (for special parameters) and Koornwinder [K] (for general parameters). In the special case of one variable ($n=1$) these polynomials reduce to monic Askey-Wilson polynomials [AW, KS]

$$p_l^{AW}(x) = \frac{(t_0t_1, t_0t_2, t_0t_3;q)_l}{t_0^l(t_0t_1t_2t_3q^{l-1};q)_l}\, {}_4\phi_3\left(\begin{matrix} q^{-l},\ t_0t_1t_2t_3q^{l-1},\ t_0e^{i\alpha x},\ t_0e^{-i\alpha x} \\ t_0t_1,\ t_0t_2,\ t_0t_3\end{matrix};q,q\right). \tag{2.8}$$

2.2. **Wilson type.** In the Wilson case the appropriate space $\mathcal{H}^W$ consists of even and permutation invariant polynomials and is spanned by the monomials

$$m_\lambda^W(x) = m_{sym,\lambda}(x_1^2, \cdots, x_n^2), \qquad \lambda \in \Lambda. \tag{2.9}$$

The inner product on $\mathcal{H}^W$ is now determined by

$$\langle m_\lambda^W, m_\mu^W \rangle_{\Delta^W} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} m_\lambda^W(x)\, \overline{m_\mu^W(x)}\, \Delta^W(x)\, dx_1 \cdots dx_n, \tag{2.10}$$

with the weight function taken to be of the form

$$\begin{aligned} \Delta^W(x) \;\; &= \prod_{\substack{1\le j<k\le n \\ \varepsilon_1,\varepsilon_2=\pm 1}} \frac{\Gamma(\nu + i(\varepsilon_1 x_j + \varepsilon_2 x_k))}{\Gamma(i(\varepsilon_1 x_j + \varepsilon_2 x_k))} \\ &\times \prod_{\substack{1\le j\le n \\ \varepsilon=\pm 1}} \frac{\Gamma(\nu_0 + i\varepsilon x_j)\Gamma(\nu_1 + i\varepsilon x_j)\Gamma(\nu_2 + i\varepsilon x_j)\Gamma(\nu_3 + i\varepsilon x_j)}{\Gamma(2i\varepsilon x_j)}. \end{aligned} \tag{2.11}$$

Here $\Gamma(\cdot)$ denotes the gamma function and the parameters are such that

$$\nu \ge 0, \qquad \mathrm{Re}(\nu_r) > 0 \;\; (r = 0, 1, 2, 3), \tag{2.12}$$

with possible non-real parameters ν_r occurring in complex conjugate pairs. For this weight function (and real parameters) the polynomials p_λ were introduced in [D3]. In the case of one variable they reduce to monic Wilson polynomials [AW, KS]

$$\begin{aligned} p_l^W(x) \;\; &= \frac{(\nu_0+\nu_1, \nu_0+\nu_2, \nu_0+\nu_3)_l}{(-1)^l(\nu_0+\nu_1+\nu_2+\nu_3+l-1)_l} \times \\ &\quad {}_4F_3\left(\begin{matrix} -l,\ \nu_0+\nu_1+\nu_2+\nu_3+l-1,\ \nu_0+ix,\ \nu_0-ix \\ \nu_0+\nu_1,\ \nu_0+\nu_2,\ \nu_0+\nu_3 \end{matrix} ; 1 \right). \end{aligned} \tag{2.13}$$

2.3. **Continuous Hahn type.** The space $\mathcal{H}^{cH}$ is very similar to that of the Wilson case but instead of only the even sector it now consists of *all* permutation invariant polynomials. The monomial basis for the space $\mathcal{H}^{cH}$ then becomes

$$m_\lambda^{cH}(x) = m_{sym,\lambda}(x_1, \cdots, x_n), \qquad \lambda \in \Lambda. \tag{2.14}$$

The inner product is of the same form as for the Wilson case

$$\langle m_\lambda^{cH}, m_\mu^{cH} \rangle_{\Delta^{cH}} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} m_\lambda^{cH}(x)\, \overline{m_\mu^{cH}(x)}\, \Delta^{cH}(x)\, dx_1 \cdots dx_n \tag{2.15}$$

but now with a weight function given by

$$\begin{aligned} \Delta^{cH}(x) \;\; &= \prod_{1\le j<k\le n} \left(\frac{\Gamma(\nu + i(x_j - x_k))}{\Gamma(i(x_j - x_k))} \frac{\Gamma(\nu + i(x_k - x_j))}{\Gamma(i(x_k - x_j))} \right) \\ &\times \prod_{1\le j\le n} \Gamma(\nu_0^+ + ix_j)\Gamma(\nu_1^+ + ix_j)\Gamma(\nu_0^- - ix_j)\Gamma(\nu_1^- - ix_j), \end{aligned} \tag{2.16}$$

where

$$\nu \geq 0, \qquad \mathrm{Re}(\nu_0^\pm), \mathrm{Re}(\nu_1^\pm) > 0, \quad \nu_0^- = \overline{\nu_0^+}, \;\; \nu_1^- = \overline{\nu_1^+}. \tag{2.17}$$

Just as in the case of Wilson type polynomials, the polynomials p_λ^{cH} corresponding to the weight function (2.16) were introduced in [D3]. For $n = 1$ they reduce to monic continuous Hahn polynomials [AW, KS]

$$\begin{aligned} p_l^{cH}(x) &= \frac{i^l\, (\nu_0^+ + \nu_0^-, \nu_0^+ + \nu_1^-)_l}{(\nu_0^+ + \nu_0^- + \nu_1^+ + \nu_1^- + l - 1)_l} \times \\ &\qquad {}_3F_2\left(\begin{matrix} -l,\ \nu_0^+ + \nu_0^- + \nu_1^+ + \nu_1^- + l - 1,\ \nu_0^+ + ix \\ \nu_0^+ + \nu_0^-,\ \nu_0^+ + \nu_1^- \end{matrix} ; 1 \right). \end{aligned} \tag{2.18}$$

2.4. **Jacobi type.** The space $\mathcal{H}^J$ and the basis $\{m_\lambda^J\}_{\lambda \in \Lambda}$ are the same as for the Askey-Wilson type. The inner product is also of the form given there (cf. (2.5)) but the weight function gets replaced by

$$\begin{aligned} \Delta^J(x) &= \prod_{1 \leq j < k \leq n} \left| \sin \frac{\alpha}{2}(x_j + x_k) \sin \frac{\alpha}{2}(x_j - x_k) \right|^{2\nu} \\ &\quad \times \prod_{1 \leq j \leq n} \left| \sin(\frac{\alpha}{2} x_j) \right|^{2\nu_0} \left| \cos(\frac{\alpha}{2} x_j) \right|^{2\nu_1}, \end{aligned} \tag{2.19}$$

with

$$\alpha > 0, \qquad \nu \geq 0, \quad \nu_0, \nu_1 > -1/2. \tag{2.20}$$

In this case the corresponding polynomials p_λ were first introduced by Vretare [V] (see also [De, BO]). For $n = 1$ they reduce to monic Jacobi polynomials [AS, KS]

$$p_l^J(x) = \frac{2^{2l}\, (\nu_0 + 1/2)_l}{(\nu_0 + \nu_1 + l)_l}\ {}_2F_1\left(\begin{matrix} -l,\ \nu_0 + \nu_1 + l \\ \nu_0 + 1/2 \end{matrix} ;\ \sin^2\left(\frac{\alpha x}{2}\right) \right). \tag{2.21}$$

Remark: In the explicit formulas for the polynomials when $n = 1$ we have used their standard representations in terms of terminating (basic) hypergeometric series [AW, KS]

$${}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} ; z \right) = \sum_{k=0}^\infty \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, \tag{2.22}$$

$${}_{s+1}\phi_s\left(\begin{matrix} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix} ; q, z \right) = \sum_{k=0}^\infty \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s; q)_k} \frac{z^k}{(q;q)_k}, \tag{2.23}$$

where we have used Pochhammer symbols and q-shifted factorials defined by

$$(a_1, \ldots, a_r)_k = (a_1)_k \cdots (a_r)_k, \quad (a_1, \ldots, a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k$$

with $(a)_0 = (a;q)_0 = 1$ and

$$(a)_k = a(a+1) \cdots (a+k-1), \;\; (a;q)_k = (1-a)(1-aq) \cdots (1-aq^{k-1})$$

for $k = 1, 2, 3, \ldots$.

3. Second order difference or differential equations

As it turns out, all families of polynomials $\{p_\lambda\}_{\lambda\in\Lambda}$ introduced in the previous section satisfy an eigenvalue equation of the form

$$D\, p_\lambda = E_\lambda\, p_\lambda, \qquad \lambda \in \Lambda, \tag{3.1}$$

where $D : \mathcal{H} \to \mathcal{H}$ denotes a certain second order difference operator (Askey-Wilson, Wilson and continuous Hahn case) or a second order differential operator (Jacobi case). Below we will list for each family the relevant operator D together with its eigenvalues E_λ, $\lambda \in \Lambda$. In each case the proof that the polynomials p_λ indeed satisfy the corresponding eigenvalue equations boils down to demonstrating that the operator $D : \mathcal{H} \to \mathcal{H}$ maps the finite-dimensional subspaces $\mathcal{H}_\lambda = \mathrm{Span}\{m_\mu\}_{\mu\in\Lambda,\mu\leq\lambda}$ into themselves (triangularity) and that it is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle_\Delta$. In other words, one has to show that

Triangularity

$$D\, m_\lambda = \sum_{\mu\in\Lambda,\mu\leq\lambda} [D]_{\lambda,\mu}\, m_\mu, \qquad \text{with} \quad [D]_{\lambda,\mu} \in \mathbb{C} \tag{3.2}$$

and that

Symmetry

$$\langle D m_\lambda, m_\mu\rangle_\Delta = \langle m_\lambda, D m_\mu\rangle_\Delta. \tag{3.3}$$

It is immediate from these two properties and the definition of the polynomial p_λ that Dp_λ lies in $\mathcal{H}_\lambda$ and is orthogonal with respect to $\langle\cdot,\cdot\rangle_\Delta$ to all monomials m_μ, $\mu \in \Lambda$ with $\mu < \lambda$. But then comparison with the definition of p_λ shows that Dp_λ must be proportional to p_λ, i.e., p_λ is an eigenfunction of D. The corresponding eigenvalue E_λ is obtained via an explicit computation of the diagonal matrix element $[D]_{\lambda,\lambda}$ in Expansion (3.2).

For the Jacobi case a proof of the second order differential equation along the above lines was given by Vretare [V]. In the Askey-Wilson case the proof was given by Macdonald [M2] and (in general) Koornwinder [K]. The proof for the Wilson and continuous Hahn case is very similar to that of the Askey-Wilson case and has been outlined in [D3].

3.1. **Askey-Wilson type.**

The second order (q-)difference operator diagonalized by the polynomials p_λ^{AW}, $\lambda \in \Lambda$, is given by

$$D^{AW} = \sum_{1\leq j\leq n} \bigl(V_j^{AW}(x)(T_{j,q} - 1) \;+\; V_{-j}^{AW}(x)(T_{j,q}^{-1} - 1) \bigr) \tag{3.4}$$

with

$$V_{\pm j}^{AW}(x) \;=\; \frac{\prod_{0\le r\le 3}(1-t_r e^{\pm i\alpha x_j})}{(1-e^{\pm 2i\alpha x_j})\,(1-q\,e^{\pm 2i\alpha x_j})} \times \prod_{1\le k\le n,\,k\neq j}\left(\frac{1-te^{i\alpha(\pm x_j+x_k)}}{1-e^{i\alpha(\pm x_j+x_k)}}\right)\left(\frac{1-te^{i\alpha(\pm x_j-x_k)}}{1-e^{i\alpha(\pm x_j-x_k)}}\right). \tag{3.5}$$

Here the operators $T_{j,q}$ act on trigonometric polynomials by means of a q-shift of the jth variable

$$(T_{j,q}f)(e^{i\alpha x_1},\ldots,e^{i\alpha x_n}) = f(e^{i\alpha x_1},\ldots,e^{i\alpha x_{j-1}},qe^{i\alpha x_j},e^{i\alpha x_{j+1}},\ldots,e^{i\alpha x_n}). \tag{3.6}$$

The eigenvalue of D^{AW} on p_λ^{AW} has the value

$$E_\lambda^{AW} = \sum_{1\le j\le n}\left(t_0t_1t_2t_3q^{-1}t^{2n-j-1}(q^{\lambda_j}-1)+t^{j-1}(q^{-\lambda_j}-1)\right). \tag{3.7}$$

Proposition 3.1 ([K]). *The multivariable Askey-Wilson polynomials* p_λ^{AW}, $\lambda\in\Lambda$ (2.2), *satisfy the second order difference equation*

$$D^{AW}p_\lambda^{AW} \;=\; E_\lambda^{AW}\,p_\lambda^{AW}. \tag{3.8}$$

For $n=1$, Equation (3.8) reduces to the second order difference equation for the one-variable Askey-Wilson polynomials [AW, KS]

$$\begin{aligned}&\frac{\prod_{0\le r\le 3}(1-t_re^{i\alpha x})}{(1-e^{2i\alpha x})\,(1-qe^{2i\alpha x})}\left(p_l^{AW}(qx)-p_l^{AW}(x)\right) \;+\\ &\frac{\prod_{0\le r\le 3}(1-t_re^{-i\alpha x})}{(1-e^{-2i\alpha x})\,(1-qe^{-2i\alpha x})}\left(p_l^{AW}(q^{-1}x)-p_l^{AW}(x)\right) \;+\\ &= \left(t_0t_1t_2t_3q^{-1}(q^l-1)+(q^{-l}-1)\right)p_l^{AW}(x).\end{aligned} \tag{3.9}$$

3.2. **Wilson type.** In the case of Wilson type polynomials the difference operator takes the form

$$D^W = \sum_{1\le j\le n}\left(V_j^W(x)(T_j-1)\;+\;V_{-j}^W(x)(T_j^{-1}-1)\right) \tag{3.10}$$

where

$$V_{\pm j}^W(x) \;=\; \frac{\prod_{0\le r\le 3}(i\nu_r\pm x_j)}{(\pm 2ix_j)\,(\pm 2ix_j-1)} \times \prod_{1\le k\le n,\,k\neq j}\left(\frac{i\nu\pm x_j+x_k}{\pm x_j+x_k}\right)\left(\frac{i\nu\pm x_j-x_k}{\pm x_j-x_k}\right) \tag{3.11}$$

and the action of T_j is given by a unit shift of the jth variable along the imaginary axis

$$(T_j f)(x_1,\ldots,x_n)=f(x_1,\ldots,x_{j-1},x_j+i,x_{j+1},\ldots,x_n). \tag{3.12}$$

The corresponding eigenvalues now read

$$E_\lambda^W=\sum_{1\le j\le n}\lambda_j\left(\lambda_j+\nu_0+\nu_1+\nu_2+\nu_3-1+2(n-j)\nu\right). \tag{3.13}$$

Proposition 3.2 ([D3]). *The multivariable Wilson polynomials p_λ^W, $\lambda\in\Lambda$ (2.2), satisfy the second order difference equation*

$$D^W p_\lambda^W \;=\; E_\lambda^W\, p_\lambda^W. \tag{3.14}$$

For $n=1$, Equation (3.14) reduces to the second order difference equation for the one-variable Wilson polynomials [KS]

$$\begin{aligned}
&\frac{\prod_{0\le r\le 3}(i\nu_r+x)}{2ix\,(2ix-1)}\left(p_l^W(x+i)-p_l^W(x)\right)\quad+ \\
&\frac{\prod_{0\le r\le 3}(i\nu_r-x)}{2ix\,(2ix+1)}\left(p_l^W(x-i)-p_l^W(x)\right) \\
&=l\,(l+\nu_0+\nu_1+\nu_2+\nu_3-1)\,p_l^W(x).
\end{aligned} \tag{3.15}$$

3.3. **Continuous Hahn type.** For the continuous Hahn type one has

$$D^{cH}=\sum_{1\le j\le n}\left(V_{j,-}^{cH}(x)(T_j-1)\;+\;V_{j,+}^{cH}(x)(T_j^{-1}-1)\right) \tag{3.16}$$

with

$$V_{j,+}^{cH}(x)\;=\;(\nu_0^++ix_j)(\nu_1^++ix_j)\prod_{1\le k\le n,\,k\ne j}\left(1+\frac{\nu}{i(x_j-x_k)}\right), \tag{3.17}$$

$$V_{j,-}^{cH}(x)\;=\;(\nu_0^--ix_j)(\nu_1^--ix_j)\prod_{1\le k\le n,\,k\ne j}\left(1-\frac{\nu}{i(x_j-x_k)}\right). \tag{3.18}$$

The action of T_j is the same as in the Wilson case (cf. (3.12)) and the eigenvalues are given by

$$E_\lambda^{cH}=\sum_{1\le j\le n}\lambda_j\left(\lambda_j+\nu_0^++\nu_1^++\nu_0^-+\nu_1^--1+2(n-j)\nu\right). \tag{3.19}$$

Proposition 3.3 ([D3]). *The multivariable continuous Hahn polynomials p_λ^{cH}, $\lambda\in\Lambda$ (2.2), satisfy the second order difference equation*

$$D^{cH}p_\lambda^{cH}\;=\;E_\lambda^{cH}\,p_\lambda^{cH}. \tag{3.20}$$

For $n=1$, Equation (3.20) reduces to the second order difference equation for the one-variable continuous Hahn polynomials [KS]

$$\begin{aligned}(\nu_0^- - ix)(\nu_1^- - ix)\left(p_l^{cH}(x+i) - p_l^{W}(x)\right) \quad + \\ (\nu_0^+ + ix)(\nu_1^+ + ix)\left(p_l^{cH}(x-i) - p_l^{cH}(x)\right) \\ = l\left(l+\nu_0^+ + \nu_1^+ + \nu_0^- + \nu_1^- - 1\right) p_l^{cH}(x).\end{aligned} \tag{3.21}$$

3.4. **Jacobi type.** In the case of multivariable Jacobi type polynomials the operator D diagonalized by p_λ is given by a second order differential operator of the form

$$\begin{aligned}D^J = -\sum_{1\leq j\leq n} \partial_j^2 \quad - \alpha \sum_{1\leq j\leq n} \left(\nu_0 \cot(\frac{\alpha x_j}{2}) - \nu_1 \tan(\frac{\alpha x_j}{2})\right)\partial_j \\ -\alpha\nu \sum_{1\leq j<k\leq n} \left(\cot(\frac{\alpha}{2}(x_j+x_k))(\partial_j+\partial_k) + \cot(\frac{\alpha}{2}(x_j-x_k))(\partial_j-\partial_k)\right)\end{aligned} \tag{3.22}$$

where $\partial_j \equiv \partial/\partial x_j$. The eigenvalue of D^J on p_λ^J takes the value

$$E_\lambda^J = \sum_{1\leq j\leq n} \lambda_j \left(\lambda_j + \nu_0 + \nu_1 + 2(n-j)\nu\right). \tag{3.23}$$

Proposition 3.4 ([V, De])**.** *The multivariable Jacobi polynomials p_λ^J, $\lambda\in\Lambda$ (2.2), satisfy the second order differential equation*

$$D^J p_\lambda^J \;=\; E_\lambda^J\, p_\lambda^J. \tag{3.24}$$

For $n=1$, Equation (3.24) reduces to the second order differential equation for the one-variable Jacobi polynomials [AS, KS]

$$\begin{aligned}-\frac{d^2 p_\lambda^J}{dx^2}(x) \quad - \quad \alpha\left(\nu_0\cot(\frac{\alpha x}{2}) - \nu_1\tan(\frac{\alpha x}{2})\right)\frac{dp_\lambda^J}{dx}(x) \\ = \quad l\,(l+\nu_0+\nu_1)\, p_l^J(x).\end{aligned} \tag{3.25}$$

4. Limit transitions

The operator D of the previous section can be used to arrive at the following useful representation for the polynomials p_λ (cf. [M2, D5, SK])

$$p_\lambda = \left(\prod_{\mu\in\Lambda,\mu<\lambda} \frac{D-E_\mu}{E_\lambda - E_\mu}\right) m_\lambda. \tag{4.1}$$

Indeed, it is not difficult to infer that the r.h.s. of (4.1) determines a polynomial satisfying the defining properties *i.* and *ii.* stated in Section 2. To this end one uses the Triangularity (3.2) and Symmetry (3.3) of D, together with the observation that in each of the concrete cases discussed above the denominators in (4.1) are nonzero

since for parameter values indicated in Section 2 one has that (see [D1, Sec. 5.2] and [SK])

$$\mu < \lambda \implies E_\mu < E_\lambda. \tag{4.2}$$

(It is immediate from the triangularity of D that the r.h.s. of (4.1) can be written as a linear combination of monomials m_μ with $\mu \leq \lambda$; that the r.h.s. is also orthogonal to all m_μ with $\mu < \lambda$ follows from the symmetry of D and the fact that the operator in the numerator—viz. $\prod_{\mu\in\Lambda,\mu<\lambda}(D - E_\mu)$—annihilates the subspace $\mathrm{Span}\{m_\mu\}_{\mu\in\Lambda,\mu<\lambda}$ in view of the Cayley-Hamilton theorem.)

Below we will use Formula (4.1) to derive limit transitions from the Askey-Wilson type to the Wilson, continuous Hahn and Jacobi type families, respectively. The transition 'Askey-Wilson → Jacobi' has already been considered before in [M2, D1, SK] and is included here mainly for the sake of completeness. It will be put to use in Section 6 when deriving a system of recurrence relations for the multivariable Jacobi type polynomials.

4.1. **Askey-Wilson→Wilson.** When studying the limit $p_\lambda^{AW} \to p_\lambda^{W}$ it is convenient to first express the multivariable Askey-Wilson polynomials in terms of a slightly modified monomial basis consisting of the functions

$$\tilde{m}_\lambda^{AW}(x) = (2/\alpha)^{2|\lambda|}\, m_{sym,\lambda}(\sin^2(\alpha x_1/2), \ldots, \sin^2(\alpha x_n/2)), \quad \lambda \in \Lambda, \tag{4.3}$$

where $|\lambda| \equiv \lambda_1 + \cdots + \lambda_n$. Notice that

$$\lim_{\alpha\to 0} \tilde{m}_\lambda^{AW}(x) = m_\lambda^{W}(x) \tag{4.4}$$

whereas the original monomials $m_\lambda^{AW}(x)$ (2.4) all reduce to constant functions in this limit. Using the relation $\sin^2(\alpha x_j/2) = 1/2 - (e^{i\alpha x_j} + e^{-i\alpha x_j})/4$ one easily infers that the bases $\{\tilde{m}_\lambda^{AW}\}_{\lambda\in\Lambda}$ and $\{m_\lambda^{AW}\}_{\lambda\in\Lambda}$ are related by a triangular transformation of the form

$$\tilde{m}_\lambda = (-1/\alpha^2)^{|\lambda|}\, m_\lambda^{AW} + \sum_{\mu\in\Lambda,\, \mu<\lambda} a_{\lambda,\mu}\, m_\mu \qquad \text{with } a_{\lambda,\mu} \in \mathbb{R}. \tag{4.5}$$

It is clear that in Formula (4.1) we may always replace the monomial basis $\{m_\lambda\}_{\lambda\in\Lambda}$ by a different basis that is related by a unitriangular transformation, since (cf. above) the operator $\prod_{\mu\in\Lambda,\mu<\lambda}(D - E_\mu)$ in the numerator of the r.h.s. annihilates the subspace $\mathrm{Span}\{m_\mu\}_{\mu\in\Lambda,\mu<\lambda}$ because of the Cayley-Hamilton theorem. Hence, by taking in account the diagonal matrix elements in the basis transformation (4.5), one sees that Formula (4.1) can rewritten in terms of $\tilde{m}_\lambda^{AW}$ as

$$p_\lambda^{AW} = (-\alpha^2)^{|\lambda|} \left(\prod_{\mu\in\Lambda,\mu<\lambda} \frac{D^{AW} - E_\mu^{AW}}{E_\lambda^{AW} - E_\mu^{AW}} \right) \tilde{m}_\lambda^{AW}. \tag{4.6}$$

If we now substitute

$$q = e^{-\alpha}, \quad t = e^{-\alpha\nu}, \quad t_r = e^{-\alpha\nu_r} \ \ (r = 0, 1, 2, 3) \tag{4.7}$$

in D^{AW} (3.4) and E_λ^W (3.7), then we have that

$$\lim_{\alpha\to 0} \alpha^{-2}\, D^{AW} = D^W, \qquad \lim_{\alpha\to 0} \alpha^{-2}\, E_\lambda^{AW} = E_\lambda^W. \tag{4.8}$$

(Notice to this end that for $q = e^{-\alpha}$ the action of $T_{j,q}$ (3.6) on trigonometric polynomials is the same as that of T_j (3.12), i.e., the action amounts to a shift of the variable x_j over an imaginary unit: $x_j \to x_j + i$. To infer then that in the limit $\alpha \to 0$ the difference operator $\alpha^{-2} D^{AW}$ formally goes to D^W boils down to checking that the coefficients of the operator converge as advertised.)

By applying the limits (4.4) and (4.8) to Formula (4.6) we end up with the following limiting relation between the multivariable Askwey-Wilson and Wilson type polynomials.

Proposition 4.1. *For Askey-Wilson parameters given by* (4.7) *one has*

$$p_\lambda^W(x) = \lim_{\alpha\to 0}\, (-1/\alpha^2)^{|\lambda|}\, p_\lambda^{AW}(x), \qquad \lambda \in \Lambda \tag{4.9}$$

(with $|\lambda| \equiv \lambda_1 + \cdots + \lambda_n$*).*

4.2. **Askey-Wilson→continuous Hahn.** Just like in the previous subsection, the derivation of the transition $p_\lambda^{AW} \to p_\lambda^{cH}$ hinges again on Formula (4.1). If we shift the variables $x_1, \ldots, x_n$ over a half period by setting

$$x_j \to x_j - \pi/(2\alpha), \qquad j = 1, \ldots, n \tag{4.10}$$

and substitute parameters in the following way

$$\begin{aligned} &q = e^{-\alpha}, && t = e^{-\alpha\nu}, \\ &t_0 = -ie^{-\alpha\nu_0^+}, && t_1 = -ie^{-\alpha\nu_1^+}, \quad t_2 = ie^{-\alpha\nu_0^-}, \quad t_3 = ie^{-\alpha\nu_1^-}, \end{aligned} \tag{4.11}$$

then the version of Formula (4.1) for the multivariable Askey-Wilson polynomials takes the form (e_j denotes the jth unit vector in the standard basis of $\mathbb{R}^n$)

$$p_\lambda^{AW}\Bigl(x - \frac{\pi}{2\alpha}(e_1 + \cdots + e_n)\Bigr) = \prod_{\mu\in\Lambda,\, \mu<\lambda} \left(\frac{\tilde{D}^{AW} - E_\mu^{AW}}{E_\lambda^{AW} - E_\mu^{AW}} \right) \tilde{m}_\lambda^{AW}(x) \tag{4.12}$$

where

$$\tilde{D}^{AW} = \sum_{1\le j\le n} \Bigl(\tilde{V}_j^{AW}(x)(T_j - 1) \ + \ \tilde{V}_{-j}^{AW}(x)(T_j^{-1} - 1) \Bigr),$$

with

$$\begin{aligned}
\tilde{V}_j^{AW}(x) &= \frac{(1+e^{-\alpha\nu_0^+}e^{i\alpha x_j})(1+e^{-\alpha\nu_1^+}e^{i\alpha x_j})(1-e^{-\alpha\nu_0^-}e^{i\alpha x_j})(1-e^{-\alpha\nu_1^-}e^{i\alpha x_j})}{(1+e^{2i\alpha x_j})\,(1+e^{-\alpha}e^{2i\alpha x_j})} \\
&\quad\times \prod_{1\le k\le n,\, k\neq j} \left(\frac{1+e^{-\alpha\nu}e^{i\alpha(x_j+x_k)}}{1+e^{i\alpha(x_j+x_k)}}\right)\left(\frac{1-e^{-\alpha\nu}e^{i\alpha(x_j-x_k)}}{1-e^{i\alpha(x_j-x_k)}}\right) \\
\tilde{V}_{-j}^{AW}(x) &= \frac{(1-e^{-\alpha\nu_0^+}e^{-i\alpha x_j})(1-e^{-\alpha\nu_1^+}e^{-i\alpha x_j})(1+e^{-\alpha\nu_0^-}e^{-i\alpha x_j})(1+e^{-\alpha\nu_1^-}e^{-i\alpha x_j})}{(1+e^{-2i\alpha x_j})\,(1+e^{-\alpha}e^{-2i\alpha x_j})} \\
&\quad\times \prod_{1\le k\le n,\, k\neq j} \left(\frac{1+e^{-\alpha\nu}e^{-i\alpha(x_j+x_k)}}{1+e^{-i\alpha(x_j+x_k)}}\right)\left(\frac{1-e^{-\alpha\nu}e^{-i\alpha(x_j-x_k)}}{1-e^{-i\alpha(x_j-x_k)}}\right)
\end{aligned}$$

and

$$\tilde{m}_\lambda^{AW}(x) \equiv m_{sym,\lambda}(2\sin(\alpha x_1),\ldots,2\sin(\alpha x_n)).$$

(Just as in the case of the transition Askey-Wilson→Wilson we have rewritten the operators $T_{j,q}$ (3.6) for $q=e^{-\alpha}$ as T_j (3.12).) After dividing by $(2\alpha)^{|\lambda|}$ the r.h.s. of (4.12) goes for $\alpha\to 0$ to the corresponding formula for the continuous Hahn polynomials (i.e. with $\tilde{D}^{AW}\to D^{cH}$, $E_\lambda^{AW}\to E_\lambda^{cH}$ and $(2\alpha)^{-|\lambda|}\tilde{m}_\lambda^{AW}\to m_\lambda^{cH}$). Hence, we now arrive at the following limiting relation between the multivariable Askey-Wilson and continuous Hahn type polynomials.

Proposition 4.2. *For Askey-Wilson parameters given by* (4.11) *one has*

$$p_\lambda^{cH}(x) = \lim_{\alpha\to 0} \frac{1}{(2\alpha)^{|\lambda|}}\, p_\lambda^{AW}\left(x-\frac{\pi}{2\alpha}\omega\right), \qquad \lambda\in\Lambda \tag{4.13}$$

where $\omega\equiv e_1+\cdots+e_n$ (with e_j denoting the jth unit vector in the standard basis of $\mathbb{R}^n$).

4.3. **Askey-Wilson→Jacobi.** To recover the Jacobi type polynomials we substitute the Askey-Wilson parameters

$$t=q^g,\quad t_0=q^{g_0},\quad t_1=-q^{g_1},\quad t_2=q^{g_0'+1/2},\quad t_3=-q^{g_1'+1/2}. \tag{4.14}$$

With these parameters the formula of the Form (4.1) for the Askey-Wilson type polynomials reduces in the limit $q\to 1$ to the corresponding formula for the Jacobi type polynomials (i.e., the difference operator D^{AW} with eigenvalues E_λ^{AW} gets replaced by the differential operator D^J with eigenvalues E_λ^J). The limit $q\to 1$ amounts to sending the difference step size to zero. In order to analyze the behavior of the operator D^{AW} for $q\to 1$ in detail it is convenient to substitute $q=e^{-\alpha\beta}$ (so the action of $T_{j,q}$ (3.6) on trigonometric polynomials amounts to the shift $x_j\to x_j+i\beta$) and then write formally $T_{j,q}=\exp(i\beta\partial_j)$. A formal expansion in β then shows that $D^{AW}\sim\beta^2 D^J$ and that $E_\lambda^{AW}\sim\beta^2 E_\lambda^J$ for $\beta\to 0$. Here the Jacobi parameters ν, ν_r

are related to the parameters g, $g_r^{(\prime)}$ in (4.14) via $\nu = g$, $\nu_0 = g_0 + g_0'$ and $\nu_1 = g_1 + g_1'$. As a consequence we obtain the following limiting relation between the multivariable Askey-Wilson and Jacobi type polynomials.

Proposition 4.3. *For Askey-Wilson parameters given by* (4.14) *one has*

$$p_\lambda^J(x) = \lim_{q\to 1} p_\lambda^{AW}(x) \tag{4.15}$$

with the Jacobi parameters ν, ν_0 and ν_1 taking the value g, $g_0 + g_0'$ and $g_1 + g_1'$, respectively.

Remarks: i. In the above derivations of Propositions 4.1, 4.2 and 4.3 we have used that the Askey-Wilson type difference operator converges formally (i.e., without specifying the domains of the operators of interest) to the corresponding operators connected with the Wilson, continuous Hahn and Jacobi type polynomials, respectively. In our case such formal limits get their precise meaning when being applied to Formula (4.1).

ii. For all our four families AW, W, cH and J the dependence of (the coefficients of) the operator D and of the eigenvalues E_λ is polynomial in the parameters t, t_0, t_1, t_2, t_3 (AW), $\nu, \nu_0, \nu_1, \nu_2, \nu_3$ (W), $\nu, \nu_0^\pm, \nu_0^\pm$ (cH) and ν, ν_0, ν_1 (J), respectively. Hence it is clear from Formula (4.1) that (the coefficients of) the polynomials p_λ are rational in these parameters. We may thus extend the parameter domains for the polynomials given in Section 2 to generic (complex) values by alternatively characterizing p_λ as the polynomial of the form $p_\lambda = m_\lambda + \sum_{\mu\in\lambda,\mu<\lambda} c_{\lambda,\mu} m_\mu$ satisfying the eigenvalue equation $Dp_\lambda = E_\lambda p_\lambda$. It is clear that the limit transitions discussed in this section then extend to these larger parameter domains of generic (complex) parameter values.

iii. In the case of one variable the limit transitions from Askey-Wilson polynomials to Wilson, continuous Hahn and Jacobi polynomials were collected in [KS] (together with many other limits between the various (basic) hypergeometric orthogonal families appearing in the (q-)Askey scheme).

5. Higher order difference or differential equations

In [D1] it was shown that the second order difference equation for the multivariable Askey-Wilson type polynomials can be extended to a system of difference equations having the structure of eigenvalue equations of the form

$$D_r\, p_\lambda = E_{r,\lambda}\, p_\lambda, \qquad r = 1, \ldots, n, \tag{5.1}$$

for n independent commuting difference operators $D_1, D_2, \ldots, D_n$ of order $2, 4, \ldots, 2n$, respectively. For $r = 1$ one recovers the second order difference equation discussed in Section 3.1. After recalling the explicit expressions for the Askey-Wilson type difference operators D_r^{AW} and their eigenvalues $E_{r,\lambda}^{AW}$, we will apply the limit transitions of Section 4 to arrive at similar systems of difference equations for the multivariable

Wilson and continuous Hahn type polynomials. In case of the transition 'Askey-Wilson→ Jacobi' the step size is sent to zero and the system of difference equations degenerates to a system of hypergeometric differential equations, thus generalizing the state of affairs for the second order operator in the previous section. This limit from Askey-Wilson type difference equations to Jacobi type differential equations has already been discussed in detail in [D1, Sec. 4], so here we will merely state the results and refrain from presenting a complete treatment of this case.

5.1. **Askey-Wilson type.** The difference operators diagonalized by the multivariable Askey-Wilson polynomials via Eq. (5.1) are given by

$$D_r^{AW} = \sum_{\substack{J\subset\{1,\ldots,n\},\, 0\leq |J|\leq r\\ \varepsilon_j=\pm 1,\, j\in J}} U^{AW}_{J^c,\, r-|J|}(x)\, V^{AW}_{\varepsilon J,\, J^c}(x)\, T_{\varepsilon J,q} \qquad r=1,\ldots,n, \tag{5.2}$$

with

$$\begin{aligned} T_{\varepsilon J,q} &= \prod_{j\in J} T_{j,q}^{\varepsilon_j} \\ V^{AW}_{\varepsilon J,\, K}(x) &= \prod_{j\in J} w^{AW}(\varepsilon_j x_j) \prod_{\substack{j,j'\in J\\ j<j'}} v^{AW}(\varepsilon_j x_j+\varepsilon_{j'}x_{j'}) v^{AW}(\varepsilon_j x_j+\varepsilon_{j'}x_{j'} - i\ln(q)/\alpha) \\ &\quad \times \prod_{\substack{j\in J\\ k\in K}} v^{AW}(\varepsilon_j x_j + x_k) v^{AW}(\varepsilon_j x_j - x_k), \\ U^{AW}_{K,p}(x) &= \\ &(-1)^p \sum_{\substack{L\subset K,\, |L|=p\\ \varepsilon_l=\pm 1,\, l\in L}} \Bigl(\prod_{l\in L} w^{AW}(\varepsilon_l x_l) \prod_{\substack{l,l'\in L\\ l<l'}} v^{AW}(\varepsilon_l x_l+\varepsilon_{l'}x_{l'}) v^{AW}(-\varepsilon_l x_l-\varepsilon_{l'}x_{l'} + i\ln(q)/\alpha) \\ &\quad \times \prod_{\substack{l\in L\\ k\in K\setminus L}} v^{AW}(\varepsilon_l x_l + x_k) v^{AW}(\varepsilon_l x_l - x_k) \Bigr), \end{aligned}$$

and

$$v^{AW}(z) = t^{-1/2}\left(\frac{1-t\,e^{i\alpha z}}{1-e^{i\alpha z}}\right) \tag{5.3}$$

$$w^{AW}(z) = (t_0t_1t_2t_3q^{-1})^{-1/2} \frac{\prod_{0\leq r\leq 3}(1-t_r\, e^{i\alpha z})}{(1-e^{2i\alpha z})(1-q\,e^{2i\alpha z})}. \tag{5.4}$$

Here the action of the operators $T^{\pm 1}_{j,q}$ is defined in accordance with (3.6). The summation in (5.2) is over all index sets $J\subset\{1,\ldots,n\}$ with cardinality $|J|\leq r$ and over all configurations of signs $\varepsilon_j\in\{+1,-1\}$ with $j\in J$. Furthermore, by convention empty products are taken to be equal to one and $U_{K,p}\equiv 1$ for $p=0$.

The corresponding eigenvalue of D_r^{AW} on p_λ^{AW} has the value

$$E_{r,\lambda}^{AW} = E_r(\tau_1 q^{\lambda_1} + \tau_1^{-1} q^{-\lambda_1}, \ldots, \tau_n q^{\lambda_n} + \tau_n^{-1} q^{-\lambda_n}; \tau_r + \tau_r^{-1}, \ldots, \tau_n + \tau_n^{-1}) \tag{5.5}$$

where

$$E_r(x_1, \ldots, x_n; y_r, \ldots, y_n) \equiv \sum_{\substack{J\subset\{1,\ldots,n\}\\ 0\leq |J|\leq r}} (-1)^{r-|J|} \prod_{j\in J} x_j \sum_{r\leq l_1\leq\cdots\leq l_{r-|J|}\leq n} y_{l_1}\cdots y_{l_{r-|J|}} \tag{5.6}$$

and

$$\tau_j = t^{n-j}(t_0t_1t_2t_3q^{-1})^{1/2}, \qquad j = 1, \ldots, n. \tag{5.7}$$

(The second sum in (5.6) is understood to be equal to 1 when $|J| = r$.)

Summarizing, we have the following theorem generalizing Proposition 3.1.

Theorem 5.1 ([D1, D4]). *The multivariable Askey-Wilson polynomials p_λ^{AW}, $\lambda \in \Lambda$ (2.2) satisfy a system of difference equations of the form*

$$D_r^{AW}\, p_\lambda^{AW} = E_{r,\lambda}^{AW}\, p_\lambda^{AW}, \qquad r = 1, \ldots, n. \tag{5.8}$$

For $r = 1$ the Difference equation (5.8) goes over in the second order Difference equation (3.8) after multiplication by a constant with value $t^{n-1}(t_0t_1t_2t_3q^{-1})^{1/2}$. More generally, one may multiply the Difference equation (5.8) for arbitrary r by the constant factor $t^{r(n-1)-r(r-1)/2}(t_0t_1t_2t_3q^{-1})^{r/2}$ to obtain a difference equation that is polynomial in the parameters t, $t_0, \ldots, t_3$ and rational in q. Such multiplication amounts to omitting the factors $t^{-1/2}$ and $(t_0t_1t_2t_3q^{-1})^{-1/2}$ in the definition of $v^{AW}(z)$ (5.3) and $w^{AW}(x)$ (5.4) and to replacing the eigenvalues by

$$\begin{aligned} E_{r,\lambda}^{AW} \quad \to \quad & t^{-r(r-1)/2} \\ & \times E_r(\tau_1^+ q^{\lambda_1} + \tau_1^- q^{-\lambda_1}, \ldots, \tau_n^+ q^{\lambda_n} + \tau_n^- q^{-\lambda_n}; \tau_r^+ + \tau_r^-, \ldots, \tau_n^+ + \tau_n^-) \end{aligned}$$

with

$$\begin{aligned} \tau_j^+ &= t^{n-1}(t_0t_1t_2t_3q^{-1})^{1/2}\, \tau_j = t_0t_1t_2t_3q^{-1}t^{2n-1-j}, \\ \tau_j^- &= t^{n-1}(t_0t_1t_2t_3q^{-1})^{1/2}\, \tau_j^{-1} = t^{j-1}. \end{aligned}$$

5.2. **Wilson type.** If we substitute Askey-Wilson parameters of the form (4.7) and divide by α^{2r}, then for $\alpha \to 0$ the operator D_r^{AW} goes over in

$$D_r^W = \sum_{\substack{J\subset\{1,\ldots,n\},\, 0\leq|J|\leq r\\ \varepsilon_j=\pm1,\, j\in J}} U_{J^c,\, r-|J|}^W(x)\, V_{\varepsilon J,\, J^c}^W(x)\, T_{\varepsilon J} \qquad r = 1, \ldots, n, \tag{5.9}$$

with

$$
\begin{aligned}
T_{\varepsilon J} &= \prod_{j\in J} T_j^{\varepsilon_j}\\
V^W_{\varepsilon J, K}(x) &= \prod_{j\in J} w^W(\varepsilon_j x_j) \prod_{\substack{j,j'\in J\\ j<j'}} v^W(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) v^W(\varepsilon_j x_j + \varepsilon_{j'} x_{j'} + i)\\
&\quad \times \prod_{\substack{j\in J\\ k\in K}} v^W(\varepsilon_j x_j + x_k) v^W(\varepsilon_j x_j - x_k),\\
U^W_{K,p}(x) &=\\
&(-1)^p \sum_{\substack{L\subset K,\, |L|=p\\ \varepsilon_l=\pm 1,\, l\in L}} \Bigl(\prod_{l\in L} w^W(\varepsilon_l x_l) \prod_{\substack{l,l'\in L\\ l<l'}} v^W(\varepsilon_l x_l + \varepsilon_{l'} x_{l'}) v^W(-\varepsilon_l x_l - \varepsilon_{l'} x_{l'} - i)\\
&\quad \times \prod_{\substack{l\in L\\ k\in K\setminus L}} v^W(\varepsilon_l x_l + x_k) v^W(\varepsilon_l x_l - x_k) \Bigr),
\end{aligned}
$$

and

$$v^W(z) = \frac{i\nu + z}{z}, \qquad w^W(z) = \frac{\prod_{0\leq r\leq 3}(i\nu_r + z)}{2iz(2iz-1)}. \tag{5.10}$$

Here the action of $T_j^{\pm}$ is taken to be in accordance with (3.12). To verify this limit it suffices to recall that for $q = e^{-\alpha}$ the action of $T_{j,q}$ (3.6) amounts to that of T_j (3.12) and to observe that for parameters (4.7) one has $\lim_{\alpha\to} v^{AW}(z) = v^W(z)$ and $\lim_{\alpha\to} \alpha^{-2} w^{AW}(z) = w^W(z)$.

The eigenvalues become in this limit

$$E^W_{r,\lambda} = E_r\left((\rho_1^W + \lambda_1)^2, \ldots, (\rho_n^W + \lambda_n)^2; (\rho_r^W)^2, \ldots, (\rho_n^W)^2\right) \tag{5.11}$$

with $E_r(\cdots;\cdots)$ taken from (5.6) and

$$\rho_j^W = (n-j)\nu + (\nu_0 + \nu_1 + \nu_2 + \nu_3 - 1)/2. \tag{5.12}$$

For the eigenvalues the computation verifying the limit is a bit more complicated than for the difference operators; it hinges on the following lemma.

Lemma 5.2 ([D1, Sec. 4.2])**.** *One has*

$$
\begin{aligned}
\lim_{\alpha\to 0} \alpha^{-2r} E_r(&e^{\alpha x_1} + e^{-\alpha x_1}, \ldots, e^{\alpha x_n} + e^{-\alpha x_n}; e^{\alpha y_r} + e^{-\alpha y_r}, \ldots, e^{\alpha y_n} + e^{-\alpha y_n})\\
&= E_r(x_1^2, \ldots, x_n^2; y_r^2, \ldots, y_n^2).
\end{aligned}
$$

We may thus conclude that the transition AW→W gives rise to the following generalization of Proposition 3.2.

Theorem 5.3. *The multivariable Wilson polynomials p_λ^W, $\lambda \in \Lambda$ (2.2) satisfy a system of difference equations of the form*

$$D_r^W \, p_\lambda^W = E_{r,\lambda}^W \, p_\lambda^W, \qquad r = 1, \ldots , n. \tag{5.13}$$

For $r = 1$ the Difference equation (5.13) coincides with the second order Difference equation (3.14).

5.3. **Continuous Hahn type.** After shifting over a half period as in (4.10) and choosing Askey-Wilson parameters of the form (4.11), the operators $\alpha^{-2r} D_r^{AW}$ go for $\alpha \to 0$ over in

$$D_r^{cH} = \sum_{\substack{J_+, J_- \subset \{1,\ldots,n\} \\ J_+ \cap J_- = \emptyset,\, |J_+|+|J_-| \leq r}} U^{cH}_{J_+^c \cap J_-^c,\, r-|J_+|-|J_-|}(x)\, V^{cH}_{J_+,J_-;\, J_+^c\cap J_-^c}(x)\, T_{J_+,\, J_-} \tag{5.14}$$

$r = 1, \ldots , n$, where

$$T_{J_+,\, J_-} = \prod_{j\in J_+} T_j^{-1} \prod_{j \in J_-} T_j$$

$$V^{cH}_{J_+,J_-;\,K}(x) = \prod_{j\in J_+} w_+^{cH}(x_j) \prod_{j\in J_-} w_-^{cH}(x_j) \prod_{j\in J_+, j'\in J_-} v^{cH}(x_j - x_{j'}) v^{cH}(x_j - x_{j'} - i)$$
$$\times \prod_{\substack{j\in J_+ \\ k\in K}} v^{cH}(x_j - x_k) \prod_{\substack{j\in J_- \\ k\in K}} v^{cH}(x_k - x_j),$$

$$U^{cH}_{K,p}(x) =$$
$$(-1)^p \sum_{\substack{L_+, L_- \subset K,\, L_+\cap L_- = \emptyset \\ |L_+|+|L_-| = p}} \Bigl(\prod_{l\in L_+} w_+^{cH}(x_l) \prod_{l\in L_-} w_-^{cH}(x_l) \prod_{\substack{l\in L_+ \\ l'\in L_-}} v^{cH}(x_l - x_{l'}) v^{cH}(x_{l'} - x_l + i)$$
$$\times \prod_{\substack{l\in L_+ \\ k\in K\setminus L_+\cup L_-}} v^{cH}(x_l - x_k) \prod_{\substack{l\in L_- \\ k\in K\setminus L_+\cup L_-}} v^{cH}(x_k - x_l) \Bigr),$$

and

$$v^{cH}(z) = (1 + \frac{\nu}{iz}), \tag{5.15}$$

$$w_+^{cH}(z) = (\nu_0^+ + iz)(\nu_1^+ + iz), \qquad w_-^{cH}(z) = (\nu_0^- - iz)(\nu_1^- - iz). \tag{5.16}$$

(The action of the operators $T_j^{\pm}$ is again in accordance with (3.12).) To verify this transition one uses that $v^{AW}(\pm(x_j + x_k)) \to 1$, $v^{AW}(x_j - x_k) \to v^{cH}(x_k - x_j)$ and that $\alpha^{-2} w^{AW}(\pm x_j) \to w_{\mp}^{cH}(x_j)$, if one sends α to zero after having substituted (4.10) and (4.11).

The computation of the limit of the eigenvalues is exactly the same as in the Wilson case and the result reads

$$E_{r,\lambda}^{cH} = E_r \left((\rho_1^{cH} + \lambda_1)^2, \ldots , (\rho_n^{cH} + \lambda_n)^2; (\rho_r^{cH})^2, \ldots , (\rho_n^{cH})^2 \right) \tag{5.17}$$

with $E_r(\cdots;\cdots)$ taken from (5.6) and

$$\rho_j^{cH} = (n-j)\nu + (\nu_0^+ + \nu_1^+ + \nu_0^- + \nu_1^- - 1)/2. \tag{5.18}$$

Hence, we arrive the following generalization of Proposition 3.3.

Theorem 5.4. *The multivariable continuous Hahn polynomials p_λ^{cH}, $\lambda \in \Lambda$ (2.2) satisfy a system of difference equations of the form*

$$D_r^{cH}\, p_\lambda^{cH} = E_{r,\lambda}^{cH}\, p_\lambda^{cH}, \qquad r = 1, \ldots, n. \tag{5.19}$$

For $r = 1$ the Difference equation (5.19) coincides with the second order Difference equation (3.20).

5.4. **Jacobi type.** Ater substituting Askey-Wilson parameters given by (4.14) and dividing by a constant with value $(1-q)^{2r}$ the rth difference equation in Theorem 5.1 for the Askey-Wilson type polynomials goes for $q \to 1$ over in a differential equation $D_r^J p_\lambda^J = E_{r,\lambda}^J p_\lambda^J$ of order $2r$ for the multivariable Jacobi type polynomials [D1, Sec. 4]. The computation of the eigenvalue $E_{r,\lambda}^J = \lim_{q\to 1}(1-q)^{-2r} E_{r,\lambda}^{AW}$ hinges again on Lemma 5.2 and the result is

$$E_{r,\lambda}^J = E_r\left((\rho_1^J + \lambda_1)^2, \ldots, (\rho_n^J + \lambda_n)^2; (\rho_r^J)^2, \ldots, (\rho_n^J)^2\right) \tag{5.20}$$

with $E_r(\cdots;\cdots)$ taken from (5.6) and

$$\rho_j^J = (n-j)\nu + (\nu_0 + \nu_1)/2, \tag{5.21}$$

where $\nu = g$, $\nu_0 = g_0 + g_0'$ and $\nu_1 = g_1 + g_1'$. For $r = 1$ the differential operator $D_r^J = \lim_{q\to 1}(1-q)^{-2r} D_r^{AW}$ is given by D^J (3.22). More generally one has that D_r^J is of the form

$$D_r^J = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=r}} \prod_{j\in J} \partial_j^2 \quad + \text{l.o.} \tag{5.22}$$

(where l.o. stands for the parts of lower order in the partials), but it seems difficult to obtain the relevant differential operators for arbitrary r in explicit form starting from D_r^{AW} (5.2).

Theorem 5.5 ([D1, Sec. 4])**.** *The multivariable Jacobi polynomials p_λ^J, $\lambda \in \Lambda$ (2.2) satisfy a system of differential equations of the form*

$$D_r^J\, p_\lambda^J = E_{r,\lambda}^J\, p_\lambda^J, \qquad r = 1, \ldots, n, \tag{5.23}$$

where $D_r^J = \lim_{q\to 1}(1-q)^{-2r} D_r^{AW}$ is of the form (5.22) *and the corresponding eigenvalues are given by* (5.20).

For $r = 1$ the Differential equation (5.23) coincides with the second order Differential equation (3.24).

Remarks: i. The proof in [D1, D4] demonstrating that the multivariable Askey-Wilson type polynomials satisfy the system of difference equations in Theorem 5.1

runs along the same lines as the proof for the special case when $r = 1$ (Proposition 3.1): it consists of demonstrating that the operator D_r^{AW} is triangular with respect to the monomial basis $\{m_\lambda^{AW}\}_{\lambda\in\Lambda}$ and that it is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle_{\Delta^{AW}}$. The triangularity proof consists of two parts. First it is shown that $D_r^{AW} m_\lambda^{AW}$ lies in the space $\mathcal{H}^{AW}$ by inferring that poles originating from the zeros in the denominators of $v^{AW}(z)$ (5.3) and $w^{AW}(z)$ (5.4) all cancel each other. Next, it is verified that the operator D_r^{AW} indeed subtriangular by analyzing the asymptotics for x at infinity.

It is noteworthy to observe that if one tries to apply the same approach to arrive at a direct proof of the system of difference equations for the multivariable Wilson and continuous Hahn type polynomials (i.e., without using the fact that these two cases may be seen as limiting cases of the Askey-Wilson type polynomials), then some complications arise. In both cases the proof that the operator D_r maps the space $\mathcal{H}$ into itself and that it is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle_\Delta$ applies without significant changes. However, it is now not so simple to deduce from the asymptotics at infinity that the difference operator D_r is indeed triangular. The reason—in a nutshell—is that the functions $w^W(z)$ (5.10) and $w_\pm^{cH}(z)$ (5.16) no longer have constant asymptotics for $z \to \infty$ (as does $w^{AW}(z)$ for $\exp(i\alpha z) \to \infty$). As a consequence, it is now much more difficult than for the Askey-Wilson case to rule out the a priori possibility that monomials m_μ with $\mu \not\leq \lambda$ enter the expansion of $D_r m_\lambda$ in monomial symmetric functions (in principle D_r might a priori raise the degrees of the polynomials). For $r = 1$ one easily checks by inspection that such m_μ with $\mu \not\leq \lambda$ indeed do not appear in the expansion of $D_r m_\lambda$ but for general r this is not so easily seen from the explicit expressions at hand.

ii. Some years ago, a rather explicit characterization of a family of commuting differential operators simultaneously diagonalized by the Jacobi type polynomials p_λ^J (and generating the same algebra as the operators $D_1^J, \ldots, D_n^J$) was presented by Debiard [De]. Alternatively, it also turned out possible to express such differential operators in terms of symmetric functions of Heckman's trigonometric generalization of the Dunkl differential-reflection operators related to the root system BC_n [H, Du]. In both cases, however, it is a nontrivial problem to deduce from those results an explicit combinatorial formula for (the coefficients of) the operator D_r^J for arbitrary r. In the case of Debiard's operators one problem is that the corresponding eigenvalues do not seem to be known precisely in closed form (and also that one would like to commute all coefficients to the left); in the case of an expression in terms of trigonometric Dunkl type differential-reflection operators it appears to be difficult to explicitly compute the differential operator corresponding to the restriction of the relevant symmetrized differential-reflection operators to the space of symmetric polynomials (except when the order of the symbol is small). In the latter case the problem is that one has to commute all reflection operators to the right (and preferably all

coefficients to the left). This poses a combinatorial exercise that seems tractable only for small order of the symbol.

iii. If one transforms the operator $D^J = D_1^J$ to a second order differential operator that is self-adjoint in $L^2(\mathbb{R}^n, dx_1\cdots dx_n)$ by conjugating with the square root of the weight function $\Delta^J(x)$ (2.19), then one arrives at a Schrödinger operator of the form

$$\begin{aligned}(\Delta^J)^{-\frac{1}{2}} D^J (\Delta^J)^{\frac{1}{2}} = \\ -\sum_{1\leq j\leq n} \partial_j^2 \; + \; \frac{1}{4}\,\alpha^2 \sum_{1\leq j\leq n} \Bigl(\frac{\nu_0(\nu_0-1)}{\sin^2(\frac{\alpha x_j}{2})} + \frac{\nu_1(\nu_1-1)}{\cos^2(\frac{\alpha x_j}{2})} \Bigr) \\ +\frac{1}{2}\,\nu(\nu-1)\,\alpha^2 \sum_{1\leq j<k\leq n} \Bigl(\frac{1}{\sin^2\frac{\alpha}{2}(x_j-x_k)} + \frac{1}{\sin^2\frac{\alpha}{2}(x_j-x_k)} \Bigr) \quad - \varepsilon_0.\end{aligned} \tag{5.24}$$

where $\varepsilon_0 = \alpha^2 \sum_{1\leq j\leq n} (\rho_j^J)^2$. Explicit expressions for n independent commuting differential operators generating the same commutative algebra as transformed operators $(\Delta^J)^{-\frac{1}{2}} D_r^J (\Delta^J)^{\frac{1}{2}}$, $r = 1, \ldots, n$ can be found in the literature as a special case of the formulas presented in [OOS, OS]. However, just as in the previous remark it is again nontrivial to determine explicitly the relation between our transformed operators $(\Delta^J)^{-\frac{1}{2}} D_r^J (\Delta^J)^{\frac{1}{2}}$, $r = 1, \ldots, n$ and the relevant specialization of the differential operators in [OOS, OS], because to our knowledge the eigenvalues of the latter operators are not available in closed form (except in cases when the order of the symbol is small).

6. Recurrence relations

In this section we first recall the system of recurrence relations for the multivariable Askey-Wilson type polynomials presented in [D5]. Next, the limit transitions of Section 4 are applied to arrive at similar systems of recurrence relations for the multivariable Wilson, continuous Hahn and Jacobi type polynomials, respectively. To describe these recurrence relations it is convenient to pass from the monic polynomials $p_\lambda(x)$ to a different normalization by introducing

$$P_\lambda(x) \equiv c_\lambda\, p_\lambda(x), \qquad\qquad c_\lambda = c^{|\lambda|}\, \frac{\hat{\Delta}_+(\rho)}{\hat{\Delta}_+(\rho+\lambda)}, \tag{6.1}$$

where c denotes some constant not depending on λ (recall also that $|\lambda| \equiv \lambda_1 + \cdots + \lambda_n$) and the function $\hat{\Delta}_+(x)$ is of the form

$$\hat{\Delta}_+(x) = \prod_{1\leq j<k\leq n} \hat{d}_{v,+}(x_j+x_k)\, \hat{d}_{v,+}(x_j-x_k) \prod_{1\leq j\leq n} \hat{d}_{w,+}(x_j). \tag{6.2}$$

The precise value of the constant c, the vector $\rho = (\rho_1, \ldots, \rho_n)$ and the form of the functions $\hat{d}_{v,+}$, $\hat{d}_{w,+}$ depends on the type of polynomials of interest and will be detailed below separately for each case.

The general structure of the recurrence relations for the renormalized polynomials $P_\lambda(x)$ reads

$$\hat{E}_r(x)\,P_\lambda(x) = \sum_{\substack{J\subset\{1,\ldots,n\},\,0\leq|J|\leq r\\ \varepsilon_j=\pm1,\,j\in J;\; e_{\varepsilon J}+\lambda\in\Lambda}} \hat{U}_{J^c,\,r-|J|}(\rho+\lambda)\,\hat{V}_{\varepsilon J,\,J^c}(\rho+\lambda)\,P_{\lambda+e_{\varepsilon J}}(x) \tag{6.3}$$

with $r=1,\ldots,n$, and

$$\begin{aligned}
e_{\varepsilon J} &= \sum_{j\in J}\varepsilon_j e_j,\\
\hat{V}_{\varepsilon J,\,K}(x) &= \prod_{j\in J}\hat{w}(\varepsilon_j x_j)\prod_{\substack{j,j'\in J\\ j<j'}}\hat{v}(\varepsilon_j x_j+\varepsilon_{j'}x_{j'})\,\hat{v}(\varepsilon_j x_j+\varepsilon_{j'}x_{j'}+1)\\
&\quad\times\prod_{\substack{j\in J\\ k\in K}}\hat{v}(\varepsilon_j x_j+x_k)\,\hat{v}(\varepsilon_j x_j-x_k),\\
\hat{U}_{K,p}(x) &= \\
&(-1)^p\sum_{\substack{L\subset K,\,|L|=p\\ \varepsilon_l=\pm1,\,l\in L}}\Bigl(\prod_{l\in L}\hat{w}(\varepsilon_l x_l)\prod_{\substack{l,l'\in L\\ l<l'}}\hat{v}(\varepsilon_l x_l+\varepsilon_{l'}x_{l'})\,\hat{v}(-\varepsilon_l x_l-\varepsilon_{l'}x_{l'}-1)\\
&\quad\times\prod_{\substack{l\in L\\ k\in K\setminus L}}\hat{v}(\varepsilon_l x_l+x_k)\,\hat{v}(\varepsilon_l x_l-x_k)\Bigr).
\end{aligned}$$

The functions $\hat{E}_1(x),\ldots,\hat{E}_n(x)$ appearing in the l.h.s. of (6.3) denote certain (explicitly given) symmetric polynomials that generate the algebra of all symmetric polynomials. The rth recurrence relation expresses the fact that the expansion of the product $\hat{E}_r(x)P_\lambda(x)$ in terms of the basis elements $P_\mu(x)$, $\mu\in\Lambda$ is known explicitly (i.e., the coefficients in the expansion are known in closed form). The expansion coefficients are determined by the functions $\hat{v}$ and $\hat{w}$ (together with the vector ρ) whose precise form again depends on the class of polynomials of interest. In combinatorics one sometimes refers to this type of recurrence relations as to generalized Pieri type formulas after similar expansion formulas for the products of elementary or complete symmetric functions and Schur functions (in terms of the latter functions) [M4]. In the simplest case, i.e. when $r=1$, the Recurrence formula (6.3) reduces to an expression of the form

$$\begin{aligned}
\hat{E}(x)P_\lambda(x) &= \sum_{\substack{1\leq j\leq n\\ \lambda+e_j\in\Lambda}}\hat{V}_j(\rho+\lambda)\bigl(P_{\lambda+e_j}(x)-P_\lambda(x)\bigr)+ \\
&\quad \sum_{\substack{1\leq j\leq n\\ \lambda-e_j\in\Lambda}}\hat{V}_{-j}(\rho+\lambda)\bigl(P_{\lambda-e_j}(x)-P_\lambda(x)\bigr)
\end{aligned} \tag{6.4}$$

with $\hat{E}(x)=\hat{E}_1(x)$ and

$$\hat{V}_{\pm j}(x)=\hat{w}(\pm x_j)\prod_{1\leq k\leq n,\, k\neq j}\hat{v}(\pm x_j+x_k)\,\hat{v}(\pm x_j-x_k). \tag{6.5}$$

We will see that for the four families considered below this formula reduces in the case of one variable to the well-known three-term recurrence relation for the Askey-Wilson, Wilson, continuous Hahn or Jacobi polynomials, respectively.

Note: To date a complete proof (contained in [D5]) of the recurrence relations for the multivariable Askey-Wilson type polynomials exists only for parameters satisfying the (self-duality) condition

$$q\,t_0\,t_1^{-1}t_2^{-1}t_3^{-1}=1. \tag{6.6}$$

It is for this reason that in the present formulation of the system of recurrence relations for our multivariable hypergeometric polynomials given below some restrictions on the parameters are imposed (except in the case of Jacobi type polynomials where such a restriction turns out to be not necessary). As is argued in [D5] for the Askey-Wilson type, however, it is expected (and easily checked for the special case of one variable) that our recurrence relations remain valid also for parameter values not meeting these restrictions (cf. Remark *i.* at the end of this section).

6.1. **Askey-Wilson type.** The normalization constants c_λ^{AW} for the polynomials in (6.1) of Askey-Wilson type $P_\lambda^{AW}(x)$ are determined by the constant $c^{AW}=1$ and the functions

$$\hat{d}_{v,+}^{AW}(z) = t^{-z/2}\frac{(q^z;q)_\infty}{(tq^z;q)_\infty}, \tag{6.7}$$

$$\hat{d}_{w,+}^{AW}(z) = (\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3q^{-1})^{-z/2}\frac{(q^{2z};\,q)_\infty}{(\hat{t}_0q^z,\,\hat{t}_1q^z,\,\hat{t}_2q^z,\,\hat{t}_3q^z;\,q)_\infty}. \tag{6.8}$$

Here we have introduced dependent parameters $\hat{t}_r$ that are related to the Askey-Wilson parameters t_r by

$$\begin{aligned}\hat{t}_0 &= (t_0t_1t_2t_3q^{-1})^{1/2},\\ \hat{t}_1 &= (t_0t_1t_2^{-1}t_3^{-1}q)^{1/2},\\ \hat{t}_2 &= (t_0t_1^{-1}t_2t_3^{-1}q)^{1/2},\\ \hat{t}_3 &= (t_0t_1^{-1}t_2^{-1}t_3q)^{1/2}.\end{aligned} \tag{6.9}$$

The vector $\rho=\rho^{AW}=(\rho_1^{AW},\ldots,\rho_n^{AW})$ has in the present case the components

$$\rho_j^{AW}={}^q\log\tau_j,\qquad \tau_j=t^{n-j}(t_0t_1t_2t_3q^{-1})^{1/2} \tag{6.10}$$

and is introduced mostly for notational convenience. In fact, the logarithm with base q entering through the components ρ_j^{AW} merely has a formal meaning and appears in our formulas always as an exponent of q.

The symmetric functions $\hat{E}_r^{AW}(x)$ multiplying $P_\lambda^{AW}(x)$ in the l.h.s. of (6.3) are given by

$$\hat{E}_r^{AW}(x) = E_r(e^{i\alpha x_1} + e^{-i\alpha x_1}, \ldots, e^{i\alpha x_n} + e^{-i\alpha x_n}; \hat{\tau}_r + \hat{\tau}_r^{-1}, \ldots, \hat{\tau}_n + \hat{\tau}_n^{-1}) \tag{6.11}$$

with $E_r(\cdots;\cdots)$ being taken from (5.6) (as usual) and with

$$\hat{\tau}_j = t^{n-j}(\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3 q^{-1})^{1/2} \tag{6.12}$$

(cf. (6.10)). The coefficients in the r.h.s. of (6.3) entering the expansion of the function $\hat{E}_r^{AW}(x)P_\lambda^{AW}(x)$ in terms of the basis elements $P_\mu^{AW}(x)$ are characterized by the functions

$$\hat{v}^{AW}(z) = t^{-1/2}\left(\frac{1-t\,q^z}{1-q^z}\right), \tag{6.13}$$

$$\hat{w}^{AW}(z) = (\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3 q^{-1})^{-1/2}\frac{\prod_{0\leq r\leq 3}(1-\hat{t}_r\,q^z)}{(1-q^{2z})(1-q^{2z+1})} \tag{6.14}$$

(cf. (5.3), (5.4)). We now have the following theorem from [D5].

Theorem 6.1 ([D5]). *The renormalized multivariable Askey-Wilson type polynomials $P_\lambda^{AW}(x)$, $\lambda\in\Lambda$ (2.2), satisfy a system of recurrence relations given by (6.3) (with $\hat{E}_r^{AW}$, $\hat{v}^{AW}$, $\hat{w}^{AW}$ and ρ^{AW} taken from (6.11), (6.13), (6.14) and (6.10), and with $r=1,\ldots,n$) for Askey-Wilson parameters subject to the condition $q\,t_0\,t_1^{-1}t_2^{-1}t_3^{-1}=1$.*

For $r=1$ one has

$$\hat{E}^{AW}(x) = \hat{E}_1^{AW}(x) = \sum_{1\leq j\leq n}\left(e^{i\alpha x_j} + e^{-i\alpha x_j} - \hat{\tau}_j - \hat{\tau}_j^{-1}\right).$$

The corresponding Recurrence formula (6.4), (6.5) coincides in the case of one variable with the three-term recurrence relation for the renormalized Askey-Wilson polynomials

$$P_l^{AW}(x) = {}_4\phi_3\left(\begin{matrix} q^{-l},\ t_0t_1t_2t_3q^{l-1},\ t_0e^{i\alpha x},\ t_0e^{-i\alpha x} \\ t_0t_1,\ t_0t_2,\ t_0t_3 \end{matrix};q,q\right), \tag{6.15}$$

which reads [AW, KS]

$$\begin{aligned}
&(2\cos(\alpha x) - t_0 - t_0^{-1})P_l^{AW}(x) = \\
&\frac{(1-t_0t_1t_2t_3q^{l-1})\prod_{1\leq r\leq 3}(1-t_0t_rq^l)}{t_0(1-t_0t_1t_2t_3q^{2l-1})(1-t_0t_1t_2t_3q^{2l})}\left(P_{l+1}^{AW}(x) - P_l^{AW}(x)\right) + \\
&\frac{t_0(1-q^l)\prod_{1\leq r<s\leq 3}(1-t_st_rq^{l-1})}{(1-t_0t_1t_2t_3q^{2l-2})(1-t_0t_1t_2t_3q^{2l-1})}\left(P_{l-1}^{AW}(x) - P_l^{AW}(x)\right).
\end{aligned} \tag{6.16}$$

It is clear that in this special situation Theorem 6.1 indeed holds without restriction on the parameters as was anticipated more generally for arbitrary r and n in [D5].

It is important to observe that, despite the appearances of square roots (and infinite products) in intermediate expressions, both the normalization constants c_λ^{AW} and the Askey-Wilson type Recurrence relations (6.3) are in the end rational in q and the parameters t, $t_0, \ldots, t_3$. For the recurrence relations this is is rather immediate from the fact that

$$\hat{v}^{AW}(\varepsilon_j(\rho_j^{AW}+\lambda_j)+\varepsilon_k(\rho_k^{AW}+\lambda_k)) = t^{-1/2}\left(\frac{1-t\tau_j^{\varepsilon_j}\tau_k^{\varepsilon_k}q^{\varepsilon_j\lambda_j+\varepsilon_K\lambda_k}}{1-\tau_j^{\varepsilon_j}\tau_k^{\varepsilon_k}q^{\varepsilon_j\lambda_j+\varepsilon_K\lambda_k}}\right) \tag{6.17}$$

$$\hat{w}^{AW}(\varepsilon_j(\rho_j^{AW}+\lambda_j)) = (\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3q^{-1})^{-1/2}\frac{\prod_{0\leq r\leq 3}(1-\hat{t}_r\,\tau^{\varepsilon_j}q^{\varepsilon_j\lambda_j})}{(1-\tau^{2\varepsilon_j}q^{2\varepsilon_j\lambda_j}(1-\tau^{2\varepsilon_j}q^{2\varepsilon_j\lambda_j+1})} \tag{6.18}$$

The point is that the combinations $\tau_j^{\varepsilon_j}\tau_k^{\varepsilon_k}$, $\hat{t}_r\,\tau^{\varepsilon_j}$ and $\tau^{2\varepsilon_j}$ are rational in t, $t_0, \ldots, t_3$ plus that the functions $\hat{v}^{AW}$ always emerge in pairs in (6.3) and $(\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3q^{-1})^{1/2} = t_0$ (hence the square root constant factors in $\hat{v}^{AW}$, $\hat{w}^{AW}$ do not spoil the rationality). The last equality is also needed to see that $\hat{E}_r^{AW}(x)$ (6.11) in the l.h.s. of the recurrence relation depends rationally on the parameters: $\hat{\tau}_j = t^{n-j}(\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3q^{-1})^{1/2} = t^{n-j}t_0$. For c_λ^{AW} the rationality in the parameters is seen similarly after rewriting in the form (by cancelling common factors in numerator and denominator)

$$\begin{aligned} c_\lambda^{AW} \;&=\; C\prod_{1\leq j<k\leq n}\frac{(\tau_j\tau_k;\,q)_{\lambda_j+\lambda_k}}{(t\tau_j\tau_k;\,q)_{\lambda_j+\lambda_k}}\frac{(\tau_j\tau_k^{-1};\,q)_{\lambda_j-\lambda_k}}{(t\tau_j\tau_k^{-1};\,q)_{\lambda_j-\lambda_k}} \\ &\quad\times\prod_{1\leq j\leq n}\frac{(\tau_j^2;\,q)_{2\lambda_j}}{(\hat{t}_0\tau_j,\,\hat{t}_1\tau_j,\,\hat{t}_2\tau_j,\,\hat{t}_3\tau_j;\,q)_{\lambda_j}} \end{aligned} \tag{6.19}$$

where $C = \prod_{1\leq j\leq n}\hat{\tau}_j^{\lambda_j}$.

6.2. **Wilson type.** If we substitute Askey-Wilson parameters in accordance with (4.7), then the polynomials $P_\lambda^{AW}(x)$ converge for $\alpha\to 0$ to renormalized multivariable Wilson type polynomials $P_\lambda^W(x) = c_\lambda^W p_\lambda(x)$. The normalization constant c_λ^W is of the form in (6.1), (6.2) with $c^W = -1$, the vector ρ^W taken from (5.12), and with the functions $\hat{d}_{v,+}^W$, $\hat{d}_{w,+}^W$ given by

$$\hat{d}_{v,+}^W(z) = \frac{\Gamma(\nu+z)}{\Gamma(z)}, \qquad\qquad \hat{d}_{w,+}^W(z) = \frac{\prod_{0\leq r\leq 3}\Gamma(\hat{\nu}_r+z)}{\Gamma(2z)}. \tag{6.20}$$

Here we have introduced dependent parameters $\hat{\nu}_r$ related to the Wilson parameters ν_r by

$$\begin{aligned} \hat{\nu}_0 &= (\nu_0+\nu_1+\nu_2+\nu_3-1)/2, \\ \hat{\nu}_1 &= (\nu_0+\nu_1-\nu_2-\nu_3+1)/2, \\ \hat{\nu}_2 &= (\nu_0-\nu_1+\nu_2-\nu_3+1)/2, \\ \hat{\nu}_3 &= (\nu_0-\nu_1-\nu_2+\nu_3+1)/2. \end{aligned} \tag{6.21}$$

To verify that the Askey-Wilson type polynomials $P_\lambda^{AW}(x)$ with parameters (4.7) indeed converge for $\alpha \to 0$ to the renormalized Wilson type polynomials $P_\lambda^W(x)$ thus defined, one uses Proposition 4.1 and the fact that (for parameters (4.7))

$$\lim_{\alpha\to 0}(-\alpha^2)^{|\lambda|} c_\lambda^{AW} = c_\lambda^W .$$

The latter limit can be easily checked with the aid of Representation (6.19) for c_λ^{AW} entailing

$$\begin{aligned}\lim_{\alpha\to 0}\, (-\alpha^2)^{|\lambda|} c_\lambda^{AW} \;&=\; (-1)^{|\lambda|} \prod_{1\leq j<k\leq n} \frac{(\rho_j^W+\rho_k^W)_{\lambda_j+\lambda_k}}{(\nu+\rho_j^W+\rho_k^W)_{\lambda_j+\lambda_k}} \frac{(\rho_j^W-\rho_k^W)_{\lambda_j-\lambda_k}}{(\nu+\rho_j^W-\rho_k^W)_{\lambda_j-\lambda_k}} \\ &\times \prod_{1\leq j\leq n} \frac{(2\rho_j^W)_{2\lambda_j}}{(\hat{\nu}_0+\rho_j^W,\, \hat{\nu}_1+\rho_j^W,\, \hat{\nu}_2+\rho_j^W,\, \hat{\nu}_3+\rho_j^W)_{\lambda_j}} .\end{aligned}$$

It is not difficult to see that the r.h.s. of this expression indeed coincides with the above defined c_λ^W of the form (6.1), (6.2) by rewriting all Pochhammer symbols as a quotient of two gamma functions.

Let us next turn to the limiting behavior of the recurrence relations. For parameters as in (4.7) and after division by α^{2r} the rth recurrence relation for the Askey-Wilson type polynomial $P_\lambda^{AW}(x)$ goes over in a recurrence relation of the form in (6.3) for the Wilson type polynomial $P_\lambda^W(x)$. The relevant symmetric function in the l.h.s. of the resulting Wilson type recurrence relation is given by

$$\hat{E}_r^W(x) = (-1)^r \sum_{\substack{J\subset\{1,\ldots,n\}\\ 0\leq|J|\leq r}} \prod_{j\in J} x_j^2 \sum_{r\leq l_1\leq\cdots\leq l_{r-|J|}\leq n} (\hat{\rho}_{l_1}^W \cdots \hat{\rho}_{l_{r-|J|}}^W)^2 \tag{6.22}$$

with

$$\hat{\rho}_j^W = (n-j)\nu + (\hat{\nu}_0+\hat{\nu}_1+\hat{\nu}_2+\hat{\nu}_3-1)/2 \tag{6.23}$$

(cf. (5.12)). Furthermore, the coefficients in the r.h.s. of this recurrence relation are determined by the vector ρ^W (5.12) and the functions

$$\hat{v}^W(z) = \frac{\nu+z}{z}, \qquad \hat{w}^W(z) = \frac{\prod_{0\leq r\leq 3}(\hat{\nu}_r+z)}{2z(2z+1)} \tag{6.24}$$

(cf. (5.10)). To check this transition from the Askey-Wilson to the Wilson type recurrence relations one uses that for Askey-Wilson parameters given by (4.7) one has

$$\lim_{\alpha\to 0}\, \alpha^{-2r} \hat{E}_r^{AW}(x) = \hat{E}_r^W(x)$$

(in view of Lemma 5.2) and that

$$\begin{aligned}&\lim_{\alpha\to 0}\, \hat{v}^{AW}(\varepsilon_j\rho_j^{AW}+\varepsilon_k\rho_k^{AW}+z) = \hat{v}^W(\varepsilon_j\rho_j^W+\varepsilon_k\rho_k^W+z),\\ &\lim_{\alpha\to 0}\, \alpha^{-2}\hat{w}^{AW}(\varepsilon_j\rho_j^{AW}+z) = \hat{w}^W(\varepsilon_j\rho_j^W+z).\end{aligned}$$

We have thus derived the following theorem.

Theorem 6.2. *The renormalized multivariable Wilson type polynomials* $P_\lambda^W(x)$, $\lambda \in \Lambda$ (2.2), *satisfy a system of recurrence relations given by* (6.3) *(with* $\hat{E}_r^W$, $\hat{v}^W$, $\hat{w}^W$ *and* ρ^W *taken from* (6.22), (6.24) *and* (5.12), *and with* $r = 1, \ldots, n$*) for Wilson parameters subject to the condition* $\nu_0 - \nu_1 - \nu_2 - \nu_3 + 1 = 0$.

The condition on the parameters in Theorem 6.2 is an immediate consequence of the parameter condition in Theorem 6.1 and may be omitted once we are sure that Theorem 6.1 holds for general parameters.

For $r = 1$ the recurrence formula is given by (6.4), (6.5) with

$$\hat{E}^W(x) = \hat{E}_1^W(x) = -\sum_{1\leq j\leq n} \left(x_j^2 + (\hat{\rho}_j^W)^2\right). \tag{6.25}$$

In the case of one variable this recurrence formula coincides with the three-term recurrence relation for the renormalized Wilson polynomials

$$P_l^W(x) = {}_4F_3\left(\begin{matrix} -l,\ \nu_0+\nu_1+\nu_2+\nu_3+l-1,\ \nu_0+ix,\ \nu_0-ix \\ \nu_0+\nu_1,\ \nu_0+\nu_2,\ \nu_0+\nu_3 \end{matrix} ;1\right), \tag{6.26}$$

which reads [KS]

$$\begin{aligned} &-(x^2+\nu_0^2)P_l^W(x) = \\ &\frac{(l+\nu_0+\nu_1+\nu_2+\nu_3-1)\prod_{1\leq r\leq 3}(l+\nu_0+\nu_r)}{(2l+\nu_0+\nu_1+\nu_2+\nu_3-1)(2l+\nu_0+\nu_1+\nu_2+\nu_3)}\left(P_{l+1}^W(x)-P_l^W(x)\right) + \\ &\frac{l\ \prod_{1\leq r<s\leq 3}(l+\nu_s+\nu_r-1)}{(2l+\nu_0+\nu_1+\nu_2+\nu_3-2)(2l+\nu_0+\nu_1+\nu_2+\nu_3-1)}\left(P_{l-1}^W(x)-P_l^W(x)\right). \end{aligned} \tag{6.27}$$

This three-term recurrence formula indeed holds without restriction on the parameters.

6.3. **Continuous Hahn type.** If we substitute Askey-Wilson parameters given by (4.11) and shift the variables x_j over a half period (cf. (4.10)), then for $\alpha \to 0$ the Askey-Wilson type polynomials $P_\lambda^{AW}(x)$ become renormalized multivariable continuous Hahn type polynomials $P_\lambda^{cH}(x) = c_\lambda^{cH} p_\lambda^{cH}(x)$ of the form in (6.1), (6.2) with $c^{cH} = 1/i$, the vector ρ^{cH} taken from (5.18), and with the functions $\hat{d}_{v,+}^{cH}$, $\hat{d}_{w,+}^{cH}$ reading

$$\hat{d}_{v,+}^{cH}(z) = \frac{\Gamma(\nu+z)}{\Gamma(z)}, \qquad \hat{d}_{w,+}^{cH}(z) = \frac{\prod_{0\leq r\leq 2}\Gamma(\hat{\nu}_r+z)}{\Gamma(2z)}. \tag{6.28}$$

Here we have again employed dependent the parameters $\hat{\nu}_0, \ldots, \hat{\nu}_3$ that are now related to the Wilson parameters $\nu_0^\pm, \nu_1^\pm$ by

$$\begin{aligned} \hat{\nu}_0 &= (\nu_0^+ + \nu_1^+ + \nu_0^- + \nu_1^- - 1)/2, \\ \hat{\nu}_1 &= (\nu_0^+ - \nu_1^+ + \nu_0^- - \nu_1^- + 1)/2, \\ \hat{\nu}_2 &= (\nu_0^+ - \nu_1^+ - \nu_0^- + \nu_1^- + 1)/2, \\ \hat{\nu}_3 &= (\nu_0^+ + \nu_1^+ - \nu_0^- - \nu_1^- + 1)/2. \end{aligned} \tag{6.29}$$

(Notice, however, that $\hat{d}_{w,+}^{cH}$ depends only on $\hat{\nu}_0, \hat{\nu}_1, \hat{\nu}_2$ and not on $\hat{\nu}_3$, which is merely introduced for convenience and will be used below (cf. (6.31)).) To verify the transition $P_\lambda^{AW}(x) \to P_\lambda^{cH}(x)$ one uses Proposition 4.2 and the fact that (for parameters (4.11)) $\lim_{\alpha\to 0}(2\alpha)^{|\lambda|} c_\lambda^{AW} = c_\lambda^{cH}$. The derivation of the limit $(2\alpha)^{|\lambda|} c_\lambda^{AW} \to c_\lambda^{cH}$ is very similar to that of the Wilson case and hinges again on Representation (6.19): first one writes c_λ^{AW} (6.19) explicitly as a rational expression in the Askey-Wilson parameters $t,\ t_0, \ldots, t_3$ by invoking of the definitions (6.9) and (6.10); next, substituting the parameters (4.11) and sending α to zero after having multiplied by $(2\alpha)^{|\lambda|}$ entails an expression for c_λ^{cH} involving Pochhammer symbols

$$\begin{aligned} c_\lambda^{cH} &= (1/i)^{|\lambda|} \prod_{1\le j<k\le n} \frac{(\rho_j^{cH} + \rho_k^{cH})_{\lambda_j+\lambda_k}}{(\nu + \rho_j^{cH} + \rho_k^{cH})_{\lambda_j+\lambda_k}} \frac{(\rho_j^{cH} - \rho_k^{cH})_{\lambda_j-\lambda_k}}{(\nu + \rho_j^{cH} - \rho_k^{cH})_{\lambda_j-\lambda_k}} \\ &\quad \times \prod_{1\le j\le n} \frac{(2\rho_j^{cH})_{2\lambda_j}}{(\hat{\nu}_0 + \rho_j^{cH},\ \hat{\nu}_1 + \rho_j^{cH},\ \hat{\nu}_2 + \rho_j^{cH})_{\lambda_j}}, \end{aligned}$$

which is seen to be equal to the stated expression of the form (6.1), (6.2) by rewriting all Pochhammer symbols as a quotient of gamma functions.

The corresponding recurrence relations for the renormalized multivariable Wilson type polynomials $P_\lambda^{cH}(x)$ are of the form given by (6.3) with the symmetric functions in the l.h.s. given by

$$\hat{E}_r^{cH}(x) = (-1)^r \sum_{\substack{J\subset\{1,\ldots,n\} \\ 0\le|J|\le r}} \prod_{j\in J} ix_j \sum_{r\le l_1\le\cdots\le l_{r-|J|}\le n} \hat{\rho}_{l_1}^{cH} \cdots \hat{\rho}_{l_{r-|J|}}^{cH} \tag{6.30}$$

with

$$\hat{\rho}_j^{cH} = (n-j)\nu + (\hat{\nu}_0 + \hat{\nu}_1 + \hat{\nu}_2 + \hat{\nu}_3 - 1)/2. \tag{6.31}$$

The coefficients in the r.h.s. are determined by the vector ρ^{cH} (5.18) and the functions

$$\hat{v}^{cH}(z) = \frac{\nu + z}{z}, \qquad \hat{w}^{cH}(z) = \frac{\prod_{0\le r\le 2}(\hat{\nu}_r + z)}{2z(2z+1)}. \tag{6.32}$$

To arrive at these recurrence relations for the multivariable continuous Hahn type polynomials we have substituted the parameters (4.11), shifted the variables over a half period (4.10) and divided both sides by $(2i\alpha)^r$. For $\alpha \to 0$ the rth Askey-Wilson

type recurrence relation then passes over in the rth continuous Hahn type recurrence relation. At this point one uses that (for parameters as in (4.11))

$$\lim_{\alpha\to 0}\ (2\alpha i)^{-r}\hat{E}_r^{AW}(x-\frac{\pi}{2\alpha}))=\hat{E}_r^{cH}(x)$$

(which immediate from the definition of $\hat{E}_r^{AW}(x)$) and that

$$\lim_{\alpha\to 0}\ \hat{v}^{AW}(\varepsilon_j\rho_j^{AW}+\varepsilon_k\rho_k^{AW}+z)=\hat{v}^{cH}(\varepsilon_j\rho_j^{cH}+\varepsilon_k\rho_k^{cH}+z),$$
$$\lim_{\alpha\to 0}\ (2i\alpha)^{-1}\hat{w}^{AW}(\varepsilon_j\rho_j^{AW}+z)=\hat{w}^{cH}(\varepsilon_j\rho_j^{cH}+z).$$

We arrive at the following theorem.

Theorem 6.3. *The renormalized multivariable continuous Hahn type polynomials $P_\lambda^{cH}(x)$, $\lambda\in\Lambda$ (2.2), satisfy a system of recurrence relations given by* (6.3) *(with $\hat{E}_r^{cH}$, $\hat{v}^{cH}$, $\hat{w}^{cH}$ and ρ^{cH} taken from* (6.30), (6.32) *and* (5.18), *and with $r=1,\ldots,n$) for continuous Hahn parameters subject to the condition $\nu_0^+-\nu_1^+-\nu_0^--\nu_1^-+1=0$.*

The condition on the continuous Hahn parameters $\nu_0^\pm,\nu_1^\pm$ in Theorem 6.3 is of course an artifact of the parameter condition in Theorem 6.1 and may again be omitted as soon as it is shown that Theorem 6.1 holds for general parameters.

For $r=1$ the recurrence relation is of the form in (6.4), (6.5) with

$$\hat{E}^{cH}(x)=\hat{E}_1^{cH}(x)=-\sum_{1\le j\le n}\left(ix_j+\hat{\rho}_j^{cH}\right). \tag{6.33}$$

In the case of one variable this formula coincides with the three-term recurrence relation for the renormalized continuous Hahn polynomials

$$P_l^{cH}(x)={}_3F_2\left(\begin{matrix}-l,\ \nu_0^++\nu_0^-+\nu_1^++\nu_1^-+l-1,\ \nu_0^++ix\\ \nu_0^++\nu_0^-,\ \nu_0^++\nu_1^-\end{matrix};1\right), \tag{6.34}$$

which reads [KS]

$$\begin{aligned}&-(ix+\nu_0^+)P_l^{cH}(x)= \\ &\frac{(l+\nu_0^++\nu_1^++\nu_0^-+\nu_1^--1)(l+\nu_0^++\nu_0^-)(l+\nu_0^++\nu_1^-)}{(2l+\nu_0^++\nu_1^++\nu_0^-+\nu_1^--1)(2l+\nu_0^++\nu_1^++\nu_0^-+\nu_1^-)}\left(P_{l+1}^{cH}(x)-P_l^{cH}(x)\right)+ \\ &\frac{-l(l+\nu_1^++\nu_0^--1)(l+\nu_1^++\nu_1^--1)}{(2l+\nu_0^++\nu_1^++\nu_0^-+\nu_1^--2)(2l+\nu_0^++\nu_1^++\nu_0^-+\nu_1^--1)}\left(P_{l-1}^{cH}(x)-P_l^{cH}(x)\right)\end{aligned} \tag{6.35}$$

and holds for general parameters.

6.4. **Jacobi type.** After substituting the Askey-Wilson parameters (4.14) and sending q to one, the Askey-Wilson type polynomials $P_\lambda^{AW}(x)$ pass over to renormalized multivariable Jacobi type polynomials $P_\lambda^J(x)=c_\lambda^J\,p_\lambda^J(x)$ of the form in (6.1), (6.2) with the Jacobi parameters ν_0 and ν_1 taking the value g_0+g_0' and g_1+g_1', respectively.

The normalization constant c_λ^J is determined by the constant $c^J = 2^{-2}$ and the vector ρ^J taken from (5.21) together with the functions $\hat{d}^J_{v,+}$ and $\hat{d}^J_{w,+}$ given by

$$\hat{d}^J_{v,+}(z) = \frac{\Gamma(\nu+z)}{\Gamma(z)}, \qquad \hat{d}^J_{w,+}(z) = \frac{\Gamma(\hat{\nu}_0+z)\Gamma(\hat{\nu}_1+z)}{\Gamma(2z)}. \tag{6.36}$$

Here we have introduced dependent parameters $\hat{\nu}_0$, $\hat{\nu}_1$ related to the Jacobi parameters ν_0, ν_1 by

$$\begin{aligned} \hat{\nu}_0 &= (\nu_0+\nu_1)/2, \\ \hat{\nu}_1 &= (\nu_0-\nu_1+1)/2. \end{aligned} \tag{6.37}$$

The verification of the transition $P_\lambda^{AW}(x) \to P_\lambda^J(x)$ hinges on Proposition 4.3 and the fact that (for parameters (4.14)) $\lim_{q\to 1} c_\lambda^{AW} = c_\lambda^J$. The latter limit is again checked using Formula (6.19) for c_λ^{AW} entailing for $q \to 1$

$$\begin{aligned} c_\lambda^J \;&=\; 2^{-2|\lambda|} \prod_{1\le j<k\le n} \frac{(\rho_j^J+\rho_k^J)_{\lambda_j+\lambda_k}}{(\nu+\rho_j^J+\rho_k^J)_{\lambda_j+\lambda_k}} \frac{(\rho_j^J-\rho_k^J)_{\lambda_j-\lambda_k}}{(\nu+\rho_j^J-\rho_k^J)_{\lambda_j-\lambda_k}} \\ &\quad \times \prod_{1\le j\le n} \frac{(2\rho_j^J)_{2\lambda_j}}{(\hat{\nu}_0+\rho_j^J,\, \hat{\nu}_1+\rho_j^J)_{\lambda_j}}, \end{aligned}$$

which seen to be equal to the stated expression for c_λ^J of the form (6.1), (6.2) by once again rewriting the Pochhammer symbols in terms of gamma functions.

The corresponding recurrence relations for $P_\lambda^J(x)$ are of the form in (6.3), with the symmetric function in the l.h.s. given by

$$\hat{E}_r^J(x) = (-1)^r \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=r}} \prod_{j\in J} \sin^2\left(\frac{\alpha x_j}{2}\right) \tag{6.38}$$

(this is $(-1)^r$ times the rth elementary symmetric function in the variables $\sin^2(\frac{\alpha x_j}{2})$, $j=1,\ldots,n$) and the coefficients in the r.h.s. being determined by ρ^J (5.21) and the functions

$$\hat{v}^J(z) = \frac{\nu+z}{z}, \qquad \hat{w}^J(z) = \frac{(\hat{\nu}_0+z)(\hat{\nu}_1+z)}{2z(2z+1)}. \tag{6.39}$$

One obtains the recurrence relations for the multivariable Jacobi type polynomials for $q\to 1$ from the Askey-Wilson type recurrence relations with parameters (4.14) if one divides both sides by 2^{2r}. To check this one uses that (for parameters given by (4.14))

$$\lim_{q\to 1} 2^{-2r}\hat{E}_r^{AW}(x) = \hat{E}_r^J(x)$$

(which is rather immediate from the definition of $\hat{E}_r^{AW}(x)$) and that

$$\lim_{q\to 1} \hat{v}^{AW}(\varepsilon_j\rho_j^{AW} + \varepsilon_k\rho_k^{AW} + z) = \hat{v}^J(\varepsilon_j\rho_j^J + \varepsilon_k\rho_k^J + z),$$
$$\lim_{q\to 1} 2^{-2}\hat{w}^{AW}(\varepsilon_j\rho_j^{AW} + z) = \hat{w}^J(\varepsilon_j\rho_j^J + z).$$

We thus arrive at the following theorem.

Theorem 6.4. *The renormalized multivariable Jacobi type polynomials $P_\lambda^J(x)$, $\lambda \in \Lambda$* (2.2)*, satisfy a system of recurrence relations given by* (6.3) *(with $\hat{E}_r^J$, $\hat{v}^J$, $\hat{w}^J$ and ρ^J taken from* (6.38), (6.39) *and* (5.21)*, and with $r = 1, \ldots, n$).*

Notice that in the Jacobi case it is not needed to impose any condition on the parameters ν_0, ν_1. The point is that if we substitute the Askey-Wilson parameters (4.14) then the condition $q\, t_0\, t_1^{-1} t_2^{-1} t_3^{-1} = 1$ in Theorem 6.1 gives rise to the condition $g_0 - g_0' - g_1 - g_1' = 0$ on the parameters g_0, g_1, g_0', g_1'. However, given this condition the Jacobi parameters $\nu_0 = g_0 + g_0'$ and $\nu_1 = g_1 + g_1'$ can still take arbitrary values. In other words: the confluence of the parameters in the limit $q \to 1$ has as consequence that the full three-parameter family of multivariable Jacobi type polynomials (parametrized by ν, ν_0 and ν_1) may be seen as limiting case of the family of (self-dual) Askey-Wilson type polynomials with parameters satisfying (6.6).

For $r = 1$ we now have a recurrence formula of the form in (6.4) with

$$\hat{E}^J(x) = \hat{E}_1^J(x) = -\sum_{1\le j\le n} \sin^2\left(\frac{\alpha x_j}{2}\right). \tag{6.40}$$

In the case of one variable this recurrence formula reduces to the three-term recurrence relation for the renormalized Jacobi polynomials

$$P_l^J(x) = {}_2F_1\left(\begin{matrix} -l,\ \nu_0+\nu_1+l \\ \nu_0+1/2 \end{matrix}\ ;\ \sin^2\left(\frac{\alpha x}{2}\right)\right) \tag{6.41}$$

reading [AS, KS] (notice, however, that our normalization of the polynomials differs slightly from the standard normalization)

$$\begin{aligned} &-\sin^2\left(\frac{\alpha x}{2}\right) P_l^J(x) = \\ &\frac{(l+\nu_0+\nu_1)(l+\nu_0+1/2)}{(2l+\nu_0+\nu_1)(2l+\nu_0+\nu_1+1)}\left(P_{l+1}^J(x) - P_l^J(x)\right) + \\ &\frac{l(l+\nu_1-1/2)}{(2l+\nu_0+\nu_1)(2l+\nu_0+\nu_1-1)}\left(P_{l-1}^J(x) - P_l^J(x)\right). \end{aligned} \tag{6.42}$$

Remarks: i. The combinatorial structure of the recurrence relations for the multivariable Askey-Wilson type polynomials is very similar to that of the difference equations in Section 5. This is by no means a coincidence. In fact, in [D5] the recurrence relations were derived from the difference equations with the aid of a

duality property for the renormalized multivariable Askey-Wilson polynomials first conjectured by Macdonald

$$P_\lambda^{AW}\left(\frac{i\ln(q)}{\alpha}(\hat{\rho}^{AW}+\mu)\right)=\hat{P}_\mu^{AW}\left(\frac{i\ln(q)}{\alpha}(\rho^{AW}+\lambda)\right). \tag{6.43}$$

Here $\hat{P}_\mu^{AW}(x)$, $\mu\in\Lambda$, denotes the renormalized multivariable Askey-Wilson type polynomial with the parameters $t_0,\ldots,t_3$ being replaced by the parameters $\hat{t}_0,\ldots,\hat{t}_3$ (cf. (6.9)) and ρ^{AW} is given by (6.10) whereas $\hat{\rho}^{AW}$ is the corresponding vector with τ_j (6.10) replaced by $\hat{\tau}_j$ (6.12). If one substitutes $x=\frac{i\ln(q)}{\alpha}(\rho^{AW}+\lambda)$ in the rth difference equation of Theorem 5.1 for the polynomial $\hat{P}_\mu^{AW}(x)$, then by using Property (6.43) (and the fact that the coefficients have a zero at $x=\frac{i\ln(q)}{\alpha}(\rho^{AW}+\lambda)$ if $\lambda+e_{\varepsilon J}\neq\Lambda$ (2.2)) one arrives at the rth recurrence relation of Theorem 6.1 (first at the points $x=\frac{i\ln(q)}{\alpha}(\hat{\rho}^{AW}+\mu)$, $\mu\in\Lambda$ and then for arbitrary x using the fact that one deals with an equality between trigonometric polynomials). In [D5] we proved Property (6.43) (and hence the recurrence relations) for parameters satisfying the self-duality condition (6.6) (implying that $\hat{t}_r=t_r$). It is of course expected, however, that Macdonald's conjecture (6.43) holds for general parameters. In the case of one variable (6.43) this is immediate from the explicit expression of the polynomials in terms of the terminating basic hypergeometric series (6.15).

ii. By applying the transition Askey-Wilson → Wilson to (6.43) one arrives at a similar duality relation for the multivariable Wilson type variables

$$P_\lambda^{W}\left(i(\hat{\rho}^{W}+\mu)\right)=\hat{P}_\mu^{W}\left(i(\rho^{W}+\lambda)\right), \tag{6.44}$$

where $\hat{P}_\mu^W(x)$ now denotes the Wilson type polynomial with the parameters $\nu_0,\ldots,\nu_3$ being replaced by $\hat{\nu}_0,\ldots,\hat{\nu}_3$ (cf. (6.21)). The self-duality condition for the parameters inherited from (6.6) then becomes $\nu_0-\nu_1-\nu_2-\nu_3+1=0$ (cf. Theorem 6.2), which implies that $\hat{\nu}_r=\nu_r$. For $n=1$ the relation (6.44) is again immediate for general parameters from the explicit expression of the polynomials in terms of the terminating hypergeometric series (6.34).

iii. For $\mu=0$ the r.h.s. of (6.43) becomes identical to one and one arrives at the relation $P_\lambda^{AW}(i\alpha^{-1}\ln\hat{\tau})=1$ (where $\ln\hat{\tau}$ stands for the vector $(\ln\hat{\tau}_1,\ldots,\ln\hat{\tau}_n)$ with $\hat{\tau}_j$ taken from (6.12)). This equality amounts to the following evaluation or specialization formula for the monic Askey-Wilson type polynomials (cf. Definition (6.1))

$$p_\lambda^{AW}(i\alpha^{-1}\ln\tau)=\frac{\hat{\Delta}_+^{AW}(\rho^{AW}+\lambda)}{\hat{\Delta}_+^{AW}(\rho^{AW})}. \tag{6.45}$$

Using the limit transitions of Section 4 one arrives at similar evaluation formulas for the multivariable Wilson, continuous Hahn and Jacobi type polynomials:

$$p_\lambda^W(i\hat{\rho}^W) = (-1)^{|\lambda|}\, \frac{\hat{\Delta}_+^W(\rho^W+\lambda)}{\hat{\Delta}_+^W(\rho^W)}, \tag{6.46}$$

$$p_\lambda^{cH}(i\hat{\rho}^{cH}) = i^{|\lambda|}\, \frac{\hat{\Delta}_+^{cH}(\rho^{cH}+\lambda)}{\hat{\Delta}_+^{cH}(\rho^{cH})}, \tag{6.47}$$

$$p_\lambda^J(0) = 2^{2|\lambda|}\, \frac{\hat{\Delta}_+^J(\rho^J+\lambda)}{\hat{\Delta}_+^J(\rho^J)}. \tag{6.48}$$

For the Askey-Wilson case this evaluation formula was conjectured by Macdonald and then proven by Cherednik [C2] for special parameters related to reduced root systems (and admissible pairs of the form $(R, R^\vee)$) with the aid of shift operators. A similar proof for the Jacobi case can be found in [Op]. The proof of the Askey-Wilson type specialization formula (6.45) based on the recurrence relations for the more general parameters subject to the condition (6.6) was first presented in [D5].

7. Orthogonality and normalization

The difference/differential equations in Section 5 hold for generic complex parameter values in view of the rational dependence on the parameters (cf. also Remark *ii.* of Section 4). The same is true for the recurrence relations in Section 6 (although we still have to impose the parameter restrictions stated in the theorems of Section 6 so as to keep our present derivation of the recurrence relations to remain valid, cf. Remark *i* of Section 6). In order to interpret the polynomials as an orthogonal system with weight function Δ, however, rather than allowing generic complex parameters we will from now on always choose the parameters from the domains given in Section 2. (The conditions on the parameters in Section 2 ensure that the relevant weight functions $\Delta(x)$ are positive and that the integrals defining the associated inner products $\langle\cdot,\cdot\rangle_\Delta$ converge in absolute value.)

It is immediate from the definition in Section 2 that the polynomial p_λ is orthogonal to p_μ for $\mu < \lambda$. It turns out that also for weights that are not comparable with respect to the partial order (2.3) the associated polynomials are orthogonal.

Theorem 7.1 (Orthogonality)**.** *The multivariable Askey-Wilson, Wilson, continuous Hahn and Jacobi type polynomials p_λ, $\lambda \in \Lambda$, form an orthogonal system with respect to the L^2 inner product with weight function $\Delta(x)$, i.e.*

$$\langle p_\lambda, p_\mu\rangle_\Delta = 0 \qquad \text{if} \qquad \lambda \neq \mu \tag{7.1}$$

(for parameters with values in the domains given in Section 2).

The proof of this theorem hinges on the following proposition.

Proposition 7.2 (Symmetry). *The operators $D_1, \ldots, D_n$ of Section 5 are symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_\Delta$, i.e.*

$$\langle D_r m_\lambda, m_\mu \rangle_\Delta = \langle m_\lambda, D_r m_\mu \rangle_\Delta \tag{7.2}$$

(for parameters with values in the domains given in Section 2).

A detailed proof of the symmetry for the Askey-Wilson type difference operators can be found in [D1, Sec. 3.4]. The same reasoning used there can also be applied to prove the proposition for the Wilson and continuous Hahn case. The Jacobi case follows from the Askey-Wilson case with the aid of the limit transition from Askey-Wilson type to Jacobi type polynomials and operators (we refer to [D1, Sec. 4] for the precise details).

By combining Proposition 7.2 with the results of Section 5 we conclude that the polynomials p_λ, $\lambda \in \Lambda$, are joint eigenfunctions of n independent operators $D_1, \ldots, D_n$ that are symmetric with respect to $\langle \cdot, \cdot \rangle_\Delta$. The corresponding (real) eigenvalues $E_{1,\lambda}, \ldots, E_{n,\lambda}$ separate the points of the integral cone Λ (2.2), i.e., if $E_{r,\lambda} = E_{r,\mu}$ for $r = 1, \ldots, n$ then λ must be equal to μ (this is seen using the fact that the functions $E_r(x_1, \ldots, x_n; y_r, \ldots y_n)$ (5.6) generate the algebra of permutation symmetric polynomials in the variables $x_1, \ldots, x_n$). It thus follows that the polynomials p_λ and p_μ with $\lambda \neq \mu$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_\Delta$ as eigenfunctions of a symmetric operator corresponding to different eigenvalues. Another proof for the orthogonality of the multivariable Askey-Wilson type polynomials was given by Koornwinder in [K]. In the case of Jacobi type polynomials independent orthogonality proofs (not using the fact that the Jacobi type is a degenerate case of the Askey-Wilson type) can be found in e.g. [De] or in [H] (upon specialization to the root system BC_n).

The main purpose of this section is to express the squared norms of the polynomials (viz. $\langle p_\lambda, p_\lambda \rangle_\Delta$) in terms of the squared norm of the unit polynomial (viz. $\langle 1, 1 \rangle_\Delta$, which corresponds to $\lambda = 0$). Our main tool to achieve this goal consists of the recurrence relations derived in the preceding section. Starting point is the identity

$$\langle \hat{E}_r P_\lambda, P_{\lambda+\omega_r} \rangle_\Delta = \langle P_\lambda, \hat{E}_r P_{\lambda+\omega_r} \rangle_\Delta, \qquad \omega_r = e_1 + \cdots + e_r, \tag{7.3}$$

where $P_\lambda(x)$ denotes the renormalized Askey-Wilson, Wilson, continuous Hahn or Jacobi type polynomial of the form (6.1) and $\hat{E}_r(x)$ is the corresponding symmetric function multiplying $P_\lambda(x)$ in the l.h.s. of the Recurrence relation (6.3). (In all four cases of interest the functions $\hat{E}_r(x)$ are real for parameters in the domains of Section 2 and we hence have (7.3) trivially.) If we work out both sides of (7.3) by replacing $\hat{E}_r P_\lambda$ and $\hat{E}_r P_{\lambda+\omega_r}$ by the corresponding r.h.s. of (6.3) and use the orthogonality of the polynomials (Theorem 7.1), then we arrive at the following relation between

$\langle P_\lambda, P_\lambda\rangle_\Delta$ and $\langle P_{\lambda+\omega_r}, p_{\lambda+\omega_r}\rangle_\Delta$

$$\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(\rho+\lambda)\langle P_{\lambda+\omega_r}, P_{\lambda+\omega_r}\rangle_\Delta = \hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(-\rho-\lambda-\omega_r)\langle P_\lambda, P_\lambda\rangle_\Delta. \tag{7.4}$$

(Recall that $\hat{U}_{K,p}=1$ for $p=0$ and that $\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}$ is taken to be in accordance with the definition in (6.3) with all signs ε_j, $j\in J$ being positive.) In principle one can use this relation to obtain for each $\lambda\in\Lambda$ the squared norm of $P_\lambda(x)$ in terms of the squared norm of $P_0(x)(=1)$ by writing λ as a positive linear combination of the (fundamental weight) vectors $\omega_1,\ldots,\omega_n$ and then apply (7.4) iteratively by walking to the weight λ starting from the zero weight $(0,\ldots,0)$ through the successive addition of fundamental weight vectors ω_r. Indeed, it is not difficult to verify that the combinatorial structure of the coefficients $\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}$ is such that the result does not depend on the order in which the fundamental weight vectors ω_r are added, i.e., the result is independent of the chosen path from $(0,\ldots,0)$ to λ. This hinges on the easily inferred (combinatorial) identity

$$\frac{\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(-x-\omega_s-\omega_r)}{\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(x+\omega_s)}\,\frac{\hat{V}_{\{1,\ldots,s\},\{s+1,\ldots,n\}}(-x-\omega_s)}{\hat{V}_{\{1,\ldots,s\},\{s+1,\ldots,n\}}(x)} = \frac{\hat{V}_{\{1,\ldots,s\},\{s+1,\ldots,n\}}(-x-\omega_r-\omega_s)}{\hat{V}_{\{1,\ldots,s\},\{s+1,\ldots,n\}}(x+\omega_r)}\,\frac{\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(-x-\omega_r)}{\hat{V}_{\{1,\ldots,r\},\{r+1,\ldots,n\}}(x)},$$

which expresses the fact that the result for the quotient of $\langle P_{\lambda+\omega_r+\omega_s}, P_{\lambda+\omega_r+\omega_s}\rangle_\Delta$ and $\langle P_\lambda, P_\lambda\rangle_\Delta$ computed via (7.4) does not depend on the order in which ω_r and ω_s are added (as it clearly should not).

To write down the answer for $\langle P_\lambda, P_\lambda\rangle_\Delta/\langle 1,1\rangle_\Delta$ for general $\lambda\in\Lambda$ we introduce the functions

$$\hat{\Delta}_\pm(x) = \prod_{1\le j<k\le n} \hat{d}_{v,\pm}(x_j+x_k)\,\hat{d}_{v,\pm}(x_j-x_k)\prod_{1\le j\le n}\hat{d}_{w,\pm}(x_j), \tag{7.5}$$

with $\hat{d}_{v,\pm}(z)$ and $\hat{d}_{w,\pm}(z)$ $(\neq 0)$ satisfying the difference equations

$$\hat{d}_{v,+}(z+1) = \hat{v}(z)\,\hat{d}_{v,+}(z), \qquad \hat{d}_{v,-}(z+1) = \hat{v}(-z-1)\,\hat{d}_{v,-}(z), \tag{7.6}$$

$$\hat{d}_{w,+}(z+1) = \hat{w}(z)\,\hat{d}_{w,+}(z), \qquad \hat{d}_{w,-}(z+1) = \hat{w}(-z-1)\,\hat{d}_{w,-}(z), \tag{7.7}$$

where $\hat{v}(z)$ and $\hat{w}(z)$ are taken to be the same as in Section 6. It is immediate from the difference equations (7.6), (7.7) that

$$\frac{\hat{\Delta}_+(x+\omega_r)}{\hat{\Delta}_+(x)} = \hat{V}_{\{1,\ldots,r\},\,\{r+1,\ldots,n\}}(x), \tag{7.8}$$

$$\frac{\hat{\Delta}_-(x+\omega_r)}{\hat{\Delta}_-(x)} = \hat{V}_{\{1,\ldots,r\},\,\{r+1,\ldots,n\}}(-x-\omega_r). \tag{7.9}$$

With the aid of the properties (7.8) and (7.9) we can rewrite Relation (7.4) in the form

$$\langle P_\lambda, P_\lambda\rangle_\Delta \, \frac{\hat{\Delta}_+(\rho+\lambda)}{\hat{\Delta}_-(\rho+\lambda)} = \langle P_{\lambda+\omega_r}, P_{\lambda+\omega_r}\rangle_\Delta \, \frac{\hat{\Delta}_+(\rho+\lambda+\omega_r)}{\hat{\Delta}_-(\rho+\lambda+\omega_r)}. \tag{7.10}$$

By using the fact that the fundamental weight vectors $\omega_1, \ldots, \omega_n$ positively generate the integral cone Λ (2.2), one deduces from this equation that the quotient $\langle P_\lambda, P_\lambda\rangle_\Delta \, \hat{\Delta}_+(\rho+\lambda)/\hat{\Delta}_-(\rho+\lambda)$ in the l.h.s. of (7.10) does not depend on the choice of $\lambda \in \Lambda$. Comparing with its evaluation in $\lambda = 0$ then entails

$$\frac{\langle P_\lambda, P_\lambda\rangle_\Delta}{\langle 1, 1\rangle_\Delta} = \frac{\hat{\Delta}_-(\rho+\lambda)\, \hat{\Delta}_+(\rho)}{\hat{\Delta}_+(\rho+\lambda)\, \hat{\Delta}_-(\rho)}. \tag{7.11}$$

As we will see below (and is suggested by the notation), it turns out that the function $\Delta_+(x)$ in the present section coincides with that of Section 6. So, by combining (7.11) with (6.1) we obtain

$$\frac{\langle p_\lambda, p_\lambda\rangle_\Delta}{\langle 1, 1\rangle_\Delta} = |c|^{-2|\lambda|} \, \frac{\hat{\Delta}_+(\rho+\lambda)\, \hat{\Delta}_-(\rho+\lambda)}{\hat{\Delta}_+(\rho)\, \hat{\Delta}_-(\rho)}. \tag{7.12}$$

We will now list for each of our four families AW, W, cH and J the associated functions $\hat{d}_{v,\pm}(z)$ and $\hat{d}_{w,\pm}(z)$, and formulate the corresponding evaluation theorem for the quotient of $\langle p_\lambda, p_\lambda\rangle_\Delta$ and $\langle 1, 1\rangle_\Delta$. The proof of this theorem boils in each case down to verifying that $\hat{d}_{v,\pm}(z)$ and $\hat{d}_{w,\pm}(z)$ indeed satisfy the difference equations (7.6) and (7.7).

In the case of multivariable Jacobi type polynomials the value of the integral for the squared norms $\langle p_\lambda, p_\lambda\rangle_\Delta$ was computed (in essence) by Opdam [Op] with the aid of shift operators (see also [HS]). Formulas for the squared norms of the Askey-Wilson type polynomials were first conjectured by Macdonald [M2, M3] and then proven by Cherednik [C1] for special parameters related to the reduced root systems (and admissible pairs of the form $(R, R^\vee)$) using a generalization of the shift operator approach. Recently, Macdonald announced a further extension of these methods to the case of general Askey-Wilson parameters [M5].

In [D5] the present author combined a proof of (7.12) using the recurrence relations along the lines sketched above with Gustafson's constant term formula [Gu, Ka] for $\langle 1, 1\rangle_{\Delta^{AW}}$, which led to an alternative derivation of Macdonald's formula for the value of $\langle p_\lambda^{AW}, p_\lambda^{AW}\rangle_{\Delta^{AW}}$ (at present in the case of parameters satisfying Condition (6.6)). The (Askey-Wilson) case is included here just for the sake of completeness. Also for the Jacobi type polynomials our proof of (7.12) with the aid of recurrence relations rather than shift operators provides, when combined with the previously derived expression for the constant term $\langle 1, 1\rangle_{\Delta^J}$ [M1, Se], an alternative way to demonstrate the validity of the known evaluation formula for $\langle p_\lambda^J, p_\lambda^J\rangle_{\Delta^J}$.

7.1. **Askey-Wilson type.** In the case of Askey-Wilson type polynomials the functions $\hat{\Delta}_\pm(x) = \hat{\Delta}^{AW}_\pm(x)$ of the form in (7.5) are characterized by $\hat{d}^{AW}_{v,+}(z)$ (6.7), $\hat{d}^{AW}_{w,+}(z)$ (6.8) and

$$\hat{d}^{AW}_{v,-}(z) = t^{z/2}\frac{(q^{z+1};q)_\infty}{(t^{-1}q^{z+1};q)_\infty}, \tag{7.13}$$

$$\hat{d}^{AW}_{w,-}(z) = (\hat{t}_0\hat{t}_1\hat{t}_2\hat{t}_3q^{-1})^{z/2}\frac{(q^{2z+1};\, q)_\infty}{(\hat{t}_0^{-1}q^{z+1},\, \hat{t}_1^{-1}q^{z+1},\, \hat{t}_2^{-1}q^{z+1},\, \hat{t}_3^{-1}q^{z+1};\, q)_\infty} \tag{7.14}$$

(with $\hat{t}_r$ given by (6.9)). Formula (7.12) leads us to the following theorem for this case.

Theorem 7.3 ([D5]). *Let us assume Askey-Wilson parameters with values taken from the domain indicated in Section 2.1 such that the recurrence relations of Theorem 6.1 hold. Then one has*

$$\frac{\langle p^{AW}_\lambda, p^{AW}_\lambda\rangle_{\Delta^{AW}}}{\langle 1,1\rangle_{\Delta^{AW}}} = \frac{\hat{\Delta}^{AW}_+(\rho^{AW}+\lambda)\,\hat{\Delta}^{AW}_-(\rho^{AW}+\lambda)}{\hat{\Delta}^{AW}_+(\rho^{AW})\,\hat{\Delta}^{AW}_-(\rho^{AW})} \tag{7.15}$$

(with ρ^{AW} given by (6.10)*).*

To complete the proof of the theorem it suffices to infer that the functions $\hat{d}^{AW}_{v,\pm}(z)$ and $\hat{d}^{AW}_{w,\pm}(z)$ indeed satisfy the corresponding difference equations of the form in (7.6) and (7.7); a fact not difficult to deduce from the standard relation for the q-shifted factorials $(a;q)_\infty = (1-a)\,(aq;q)_\infty$.

For $n = 1$ the r.h.s. of (7.15) reduces to an expression of the form

$$\langle p^{AW}_l, p^{AW}_l\rangle_{\Delta^{AW}} / \langle p^{AW}_0, p^{AW}_0\rangle_{\Delta^{AW}}$$

with (cf. [AW, KS] and recall our normalization in (2.8))

$$\langle p^{AW}_l, p^{AW}_l\rangle_{\Delta^{AW}} = \frac{4\pi\alpha^{-1}\,(t_0t_1t_2t_3q^{2l};q)_\infty}{(t_0t_1t_2t_3q^{l-1};q)_l\,(q^{l+1};q)_\infty\,\prod_{0\le r<s\le 3}(t_rt_sq^l;q)_\infty}. \tag{7.16}$$

7.2. **Wilson type.** The relevant functions $\hat{\Delta}_\pm(x) = \hat{\Delta}^W_\pm(x)$ of the form in (7.5) are determined by $\hat{d}^W_{v,+}(z)$, $\hat{d}^W_{w,+}(z)$ (6.20) and

$$\hat{d}^W_{v,-}(z) = \frac{\Gamma(-\nu+z+1)}{\Gamma(z+1)}, \qquad \hat{d}^W_{w,-}(z) = \frac{\prod_{0\le r\le 3}\Gamma(-\hat{\nu}_r+z+1)}{\Gamma(2z+1)} \tag{7.17}$$

(with $\hat{\nu}_r$ given by (6.21)). The difference equations of the form in (7.6) and (7.7) are easily verified using the standard functional equation $\Gamma(z+1) = z\Gamma(z)$ for the gamma function. The normalization theorem becomes in the present situation.

Theorem 7.4. *Let us assume Wilson parameters with values taken from the domain indicated in Section 2.2 such that the recurrence relations of Theorem 6.2 hold. Then one has*

$$\frac{\langle p_\lambda^W, p_\lambda^W\rangle_{\Delta^W}}{\langle 1,1\rangle_{\Delta^W}} = \frac{\hat{\Delta}_+^W(\rho^W+\lambda)\,\hat{\Delta}_-^W(\rho^W+\lambda)}{\hat{\Delta}_+^W(\rho^W)\,\hat{\Delta}_-^W(\rho^W)} \tag{7.18}$$

(with ρ^W given by (5.12)*).*

For $n=1$ the r.h.s. of (7.18) reduces to

$$\langle p_l^W, p_l^W\rangle_{\Delta^W}/\langle p_0^W, p_0^W\rangle_{\Delta^W}$$

with (cf. [KS] and recall our normalization in (2.13))

$$\langle p_l^W, p_l^W\rangle_{\Delta^W} = \frac{4\pi\; l!\;\prod_{0\le r<s\le 3}\Gamma(\nu_r+\nu_s+l)}{(\nu_0+\nu_1+\nu_2+\nu_3+l-1)_l\,\Gamma(\nu_0+\nu_1+\nu_2+\nu_3+2l)}. \tag{7.19}$$

7.3. **Continuous Hahn type.** The functions $\hat{\Delta}_\pm(x)=\hat{\Delta}_\pm^{cH}(x)$ of the form in (7.5) are now determined by $\hat{d}_{v,+}^{cH}(z)$, $\hat{d}_{w,+}^{cH}(z)$ (6.28) and

$$\hat{d}_{v,-}^{cH}(z) = \frac{\Gamma(-\nu+z+1)}{\Gamma(z+1)}, \qquad \hat{d}_{w,-}^{cH}(z) = \frac{\prod_{0\le r\le 2}\Gamma(-\hat{\nu}_r+z+1)}{\Gamma(2z+1)} \tag{7.20}$$

(with $\hat{\nu}_r$ taken from (6.29)). The difference equations (7.6), (7.7) follow again from the standard difference equation for the gamma function. The normalization theorem reads in this case.

Theorem 7.5. *Let us assume continuous Hahn parameters with values taken from the domain indicated in Section 2.3 such that the recurrence relations of Theorem 6.3 hold. Then one has*

$$\frac{\langle p_\lambda^{cH}, p_\lambda^{cH}\rangle_{\Delta^{cH}}}{\langle 1,1\rangle_{\Delta^{cH}}} = \frac{\hat{\Delta}_+^{cH}(\rho^{cH}+\lambda)\,\hat{\Delta}_-^{cH}(\rho^{cH}+\lambda)}{\hat{\Delta}_+^{cH}(\rho^{cH})\,\hat{\Delta}_-^{cH}(\rho^{cH})} \tag{7.21}$$

(with ρ^{cH} given by (5.18)*).*

For $n=1$ the r.h.s. of (7.21) becomes

$$\langle p_l^{cH}, p_l^{cH}\rangle_{\Delta^{cH}}/\langle p_0^{cH}, p_0^{cH}\rangle_{\Delta^{cH}}$$

with (cf. [KS] and recall our normalization in (2.18))

$$\langle p_l^{cH}, p_l^{cH}\rangle_{\Delta^{cH}} = \tag{7.22}$$

$$\frac{2\pi\; l!\;\prod_{r,s\in\{0,1\}}\Gamma(\nu_r^++\nu_s^-+l)}{(\nu_0^++\nu_0^-+\nu_1^++\nu_1^-+l-1)_l\,\Gamma(\nu_0^++\nu_0^-+\nu_1^++\nu_1^-+2l)}.$$

7.4. **Jacobi type.** For the Jacobi type polynomials the functions $\hat{\Delta}_\pm(x) = \hat{\Delta}^J_\pm(x)$ of the form in (7.5) are determined by $\hat{d}^J_{v,+}(z)$, $\hat{d}^J_{w,+}(z)$ (6.36) and

$$\hat{d}^J_{v,-}(z) = \frac{\Gamma(-\nu + z + 1)}{\Gamma(z+1)}, \qquad \hat{d}^J_{w,-}(z) = \frac{\Gamma(-\hat{\nu}_0 + z + 1)\Gamma(-\hat{\nu}_1 + z + 1)}{\Gamma(2z+1)}. \tag{7.23}$$

(with $\hat{\nu}_0$, $\hat{\nu}_1$ taken from (6.37)). The difference equations (7.6), (7.7) follow again from the difference equation for the gamma function. The normalization theorem reads in this case.

Theorem 7.6. *Let us assume Jacobi parameters with values taken from the domain indicated in Section 2.4. Then one has*

$$\frac{\langle p^J_\lambda, p^J_\lambda\rangle_{\Delta^J}}{\langle 1, 1\rangle_{\Delta^J}} = 2^{4|\lambda|}\, \frac{\hat{\Delta}^J_+(\rho^J + \lambda)\, \hat{\Delta}^J_-(\rho^J + \lambda)}{\hat{\Delta}^J_+(\rho^J)\, \hat{\Delta}^J_-(\rho^J)} \tag{7.24}$$

(with ρ^J given by (5.21)).

For $n = 1$ the r.h.s. of (7.24) becomes

$$\langle p^J_l, p^J_l\rangle_{\Delta^J} / \langle p^J_0, p^J_0\rangle_{\Delta^J}$$

with (cf. [AS, KS] and recall our normalization in (2.21))

$$\langle p^J_l, p^J_l\rangle_{\Delta^J} = \frac{2^{4l+1}\alpha^{-1}\, l!\, \Gamma(\nu_0 + 1/2)\Gamma(\nu_1 + 1/2)}{(\nu_0 + \nu_1 + l)_l\, \Gamma(\nu_0 + \nu_1 + 2l + 1)}. \tag{7.25}$$

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Department of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153, Japan